Piecewise Polynomial Collocation for the Double Layer Potential Equation over Polyhedral Boundaries

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1 INTRODUCTION

One way to solve boundary value problems for elliptic partial differential equations consists in reducing the equation over the domain to an integral equation over the boundary. For any boundary value problem, there exist various versions of boundary integral formulations which are equivalent to the original problem. Among these variants the boundary integral equations with strongly elliptic integral operators seem to be the most popular ones since the analysis of the Galerkin procedure is quite easy in this case and the matrix of the arising linear system is self adjoint (cf. e.g. [12, 36]). On the other hand, these boundary integral equations are often pseudo-differential equations of order different from zero. The condition numbers of linear systems arising from the discretization of such equations tend to infinity when the mesh size tends to zero. To solve these systems, certain kind of multigrid methods or general iterative methods together with special preconditioners are needed (cf. [20, 27, 26, 8, 13]). Therefore, it may be preferable to consider zero order boundary integral equations.

For instance, the Dirichlet problem for Laplace's equation in a bounded and simply connected polyhedron $\Omega \subseteq \mathbb{R}^3$ or the Neumann problem for the same equation on $\mathbb{R}^3 \setminus \Omega$ can be reduced to the second kind integral equation $Ax = y$ over the boundary $S := \partial\Omega$ (cf. e. g. [25]), where $A = I + 2W_S$ and

$$W_Sx(Q) := [1/2 - d_\Omega(Q)]x(Q) + \frac{1}{4\pi}\int_S \frac{n_P(Q-P)}{|P-Q|^3}x(P)d_PS, \tag{1.1}$$

$$d_\Omega(Q) := \lim_{\epsilon\to 0}\frac{|\{P\in\Omega : |P-Q|<\epsilon\}|}{|\{P\in\mathbb{R}^3 : |P-Q|<\epsilon\}|}.$$

Here n_P denotes the unit vector of the interior normal to Ω at P and $|Z|$ is the Lebesgue measure of Z for any $Z\subseteq\mathbb{R}^3$. Note that, since the boundary S is not smooth, W_S is not compact. Moreover, in general, A is not strongly elliptic. Though the kernel function

$$k(Q,P) := \frac{1}{4\pi}\frac{n_P(Q-P)}{|P-Q|^3} \tag{1.2}$$

vanishes for P and Q located on the same face of S, it is of order $O(|P-Q|^{-2})$ if P and Q tend to an edge point but remain on different faces.

The Galerkin method using piecewise polynomials or similar trial functions for the solution of our equation $Ax = y$ has been considered by Elschner [18] (even an h-p version in [16]) and by Dahlberg, Verchota [14], Adolfsson, Jawerth, Goldberg, Lennerstad [1] (even for Lipschitz boundaries). For the computation of the entries of the corresponding stiffness matrix, one can apply the results of e.g. Schwab, Wendland, and Sauters [33, 34]. However, the stability analysis for the discretized Galerkin method seems not to be done yet. The analysis of the collocation methods has started already in the sixties. Convergence results under different restrictions are due to Wendland, Kleinman [35, 22], Kral [23, 3], Atkinson, Chien, and Yang [4, 5, 37]. These authors have also suggested quadratures for the computation of the integrals in the stiffness matrix. The first stability analysis for a fully discrete method, however, is due to the author who has considered Nyström's method in [31, 30]. It has turned out that quadrature methods based on one global quadrature rule show a poor rate of convergence if this rate is considered in dependence of the number of degrees of freedom. Therefore, one should rather apply discretized collocation methods, where the quadrature rules for the integrals in the stiffness matrix depend on the collocation points.

The aim of this paper is to define a fully discretized collocation method (cf. Sect.2), where the trial functions are tensor products of smoothest splines. For simplicity we shall restrict ourselves to cubic splines. These ansatz functions are defined by an exponential parametrization, i.e. the trial functions are introduced over a grid with a certain kind of geometric mesh refinement near the corner and edge points of S. We shall establish the stability of our method with respect to the spaces $L^2(S)$ and $C(S)$ in Sects.3 and 4. For the L^2 stability, we shall need an assumption on the stability of a certain finite section method (cf. the corresponding conditions in [18, 31, 30]) which is fulfilled whenever the sufficient assumptions of [35, 23, 5] are satisfied. In order to prove stability in the supremum norm, we shall introduce a slight modification of our method and, using this, we need no additional assumptions. This modification corresponds to the modifications of finite section type in [2, 9, 24, 15, 18] or to the "modification" in [14, 1] obtained by approximating the boundary. In Sect.5 we shall derive the asymptotic rates of convergence. These rates are optimal with respect to the degree of trial functions if we neglect powers of the logarithm.

Finally, let us remark that an important feature of our method is that the set of trial functions has a basis of wavelet functions (cf. [10, 11, 29]). This fact can be used to compute a wavelet compression of the stiffness matrix and to create a fast algorithm for the solution of the linear system of equations (cf. [6, 13]).

2 THE DISCRETIZED COLLOCATION

2.1 The grid

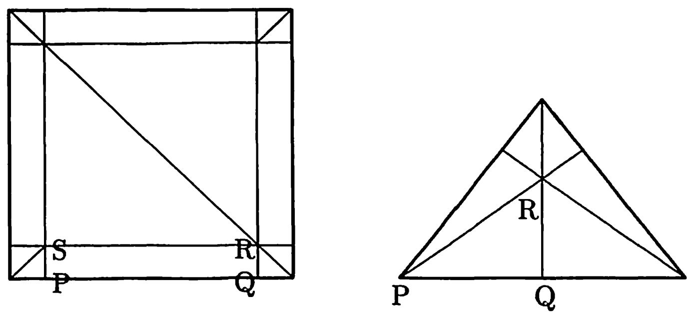

Figure 1: Partitions of a square and of a triangle.

We start with a coarse partition of S into rectangles and triangles. For this partition $S = \cup_{j=1}^{J} S^j$, we suppose that the intersection of any two subdomains is contained in a straight line and that there are four different kinds of subdomains:

a) The subdomain S^j is a triangle $\triangle PQR$, where P is the only vertex of S contained in S^j, the straight line $\overline{PQ}$ is on an edge of S, and all the other points of S^j belong to the interior of a face of S.

b) The subdomain S^j is a triangle $\triangle PQR$, where P is the only vertex of S contained in S^j and all the other points of S^j belong to the interior of a face of S.

c) The subdomain S^j is a rectangle $\square PQRS$, where S^j contains no vertex of S, the straight line $\overline{PQ}$ is on an edge of S, and all the other points of S^j belong to the interior of a face of S.

d) The subdomain S^j is a triangle $\triangle PQR$, where all points of S^j belong to the interior of a face of S.

Finally, if $S^{j_1} \cap S^{j_2}$ is contained in a straight line e, then we suppose $S^{j_1} \cap S^{j_2} = S^{j_1} \cap e = e \cap S^{j_2}$. Possible partitions for a face of S are indicated in Fig.1.

Now we introduce a parametrization $\Phi^j : D^j \longrightarrow S^j$ by

$$D^j := \begin{cases} [-\infty, 0] \times [-\infty, 0] & \text{in case a)} \\ [-\infty, 0] \times [0, 1] & \text{in case b)} \\ [0, 1] \times [-\infty, 0] & \text{in case c)} \\ [0, 1] \times [0, 1] & \text{in case d)} \end{cases}, \tag{2.1}$$

$$\Phi^j(s,t) := \begin{cases} P + e^s\,\overrightarrow{PQ} + e^s e^t\,\overrightarrow{QR} & \text{in case a)} \\ P + e^s\,\overrightarrow{PQ} + e^s t\,\overrightarrow{QR} & \text{in case b)} \\ P + s\,\overrightarrow{PQ} + e^t\,\overrightarrow{QR} & \text{in case c)} \\ P + s\,\overrightarrow{PQ} + st\,\overrightarrow{QR} & \text{in case d).} \end{cases}$$

To obtain the set of collocation points, we choose a parameter $\zeta > 0$ and a positive integer j_{lev}. We set (cf.Fig.2)

$$s_k := \begin{cases} 0 & if \;\; k = 0 \\ -h/2 & if \;\; k = 1 \\ -(k-1)h & if \;\; k = 2, \dots, M_\infty - 1 \\ -\infty & if \;\; k = M_\infty \end{cases}, \quad h = 2^{-j_{lev}}, M_\infty := \left[\zeta j_{lev} 2^{j_{lev}}\right], \quad (2.2)$$

$$t_k := \begin{cases} 0 & if \;\; k = 0 \\ h/2 & if \;\; k = 1 \\ (k-1)h & if \;\; k = 2, \dots, M_0 - 2 \\ 1 - h/2 & if \;\; k = M_0 - 1 \\ 1 & if \;\; k = M_0 \end{cases}, \quad M_0 := 2^{j_{lev}} + 2.$$

Then the collocation points on S^j are given by $P_{j,k,l} = \Phi^j(s_k, s_l)$, $k = 0, \dots, M_\infty$, $l = 0, \dots, M_\infty$ for case a), by $P_{j,k,l} = \Phi^j(s_k, t_l)$, $k = 0, \dots, M_\infty$, $l = 0, \dots, M_0$ for case b), by $P_{j,k,l} = \Phi^j(t_k, s_l)$, $k = 0, \dots, M_0$, $l = 0, \dots, M_\infty$ for case c), and by $P_{j,k,l} = \Phi^j(t_k, t_l)$, $k = 0, \dots, M_0$, $l = 0, \dots, M_0$ for case d). Note that from the definition of Φ^j it is clear that the grid $\{P_{j,k,l}\}$ is graded towards the edges and towards the vertices of S. The meshsize is of order $O(h)$. Let us collect the multiindices $\iota = (j,k,l)$ in the index set $\mathscr{I} := \{\iota\}$ and set $P_\iota := P_{j,k,l}$. In the next subsection we shall introduce piecewise continuous trial functions which are continuous on each subdomain S^j (i.e. with finite limits at the boundary points of S^j). Additionally, they are continuous at the vertices of S and at those edge points of S which are no corner points of an S^j of type c). The trial functions may have jumps at the boundary points of the subdomains S^j which do not belong to an edge of S or at corner points of the S^j of type c). Therefore, let us identify the indices of $\mathscr{I}$ belonging to one and the same vertex or edge point of $S = \partial\Omega$ which are no corner points of an S^j of type c). For points $P_{\iota_1} = P_{j_1,k_1,l_1} = P_{j_2,k_2,l_2} = P_{\iota_2}$ which do not belong to any edge of S, we distinguish between P_{ι_1} and P_{ι_2} and, for a piecewise continuous function f, we let $f(P_{\iota_1})$ and $f(P_{\iota_2})$ stand for the limit of f taken from S^{j_1} and S^{j_2}, respectively. If $P_{\iota_1} = P_{j_1,k_1,l_1} = P_{j_2,k_2,l_2} = P_{\iota_2}$ is a point at an edge e of S and a corner point of an S^j of kind c), then we distinguish between P_{ι_1} and P_{ι_2} and, for a piecewise continuous function f, we let $f(P_{\iota_1})$ and $f(P_{\iota_2})$ stand for the limit of f taken from $S^{j_1} \cap e$ and $S^{j_2} \cap e$, respectively. After these identifications, the number of collocation points $\#\mathscr{I}$ will be denoted by N and is of order $O(\zeta^2 j_{lev}^2 2^{2j_{lev}})$.

REMARK 2.1 Unfortunately, we cannot prove stability for collocation based only on the parametrizations in (2.1). The problem is that we are not able to show local stability in the vicinity of the corner points P and Q of the type c) subdomains S^j (cf. Sect.4.1). To overcome this difficulty we can introduce new parametrizations in the S^j of type a) and c) which fit together at the points P and Q of the type c) subdomains S^j. This means that, for intersecting subdomains S^{j_1} and S^{j_2} of type a) or c) with $S^{j_1} \cap S^{j_2} = \overline{QR}$

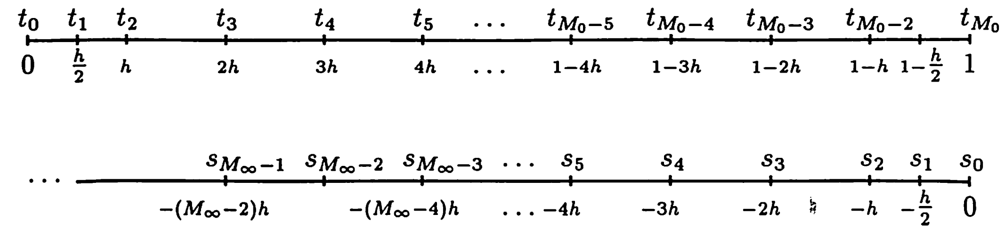

Figure 2: Grid points on $(-\infty, 0]$ and $[0, 1]$.

such that Q is an edge point, we suppose that the parametrizations and all their partial derivatives coincide at the intersections of a neighborhood of Q with $\overline{QR}$.
For example we give a parametrization for S^j of type a) which can easily be extended through $\overline{QR}$ to a parametrization (2.1) of type c). Therefore, let $\chi^1 : [-\infty, 0] \longrightarrow [0, 1]$ denote a smooth function which is identically equal to one on $[-1/4, 0]$ and which vanishes on $[-\infty, -1/2]$. We set $\chi^\rho(t) := \chi^1(\rho t)$, $\rho > 1$ and define $\Phi^j : D^j \longrightarrow S^j$ by

$$\begin{aligned}\Phi^j(s,t) \ &:= \ P + \{\chi^\rho(s)(1+s) + [1-\chi^\rho(s)]e^s\} \ \overrightarrow{PQ} + \\ &\quad \{\chi^\rho(s)\langle\chi^1(t)(1+s)e^t + [1-\chi^1(t)]e^t\rangle + [1-\chi^\rho(s)]e^s e^t\} \ \overrightarrow{QR} \, .\end{aligned} \tag{2.3}$$

For this parametrization and sufficiently large $\rho > 1$, it is easy to see that $[-\infty, 0] \ni s \mapsto \{\chi^\rho(s)(1+s) + [1-\chi^\rho(s)]e^s\}$ is a strictly increasing function mapping onto $[0, 1]$. Moreover, for sufficiently large $\rho > 1$ and for any fixed s, the function $[-\infty, 0] \ni t \mapsto \{\chi^\rho(s)\langle\chi^1(t)(1+s)e^t + [1-\chi^1(t)]e^t\rangle + [1-\chi^\rho(s)]e^s e^t\}$ maps onto $[0, \omega]$, $\omega := \{\chi^\rho(s)(1+s) + [1-\chi^\rho(s)]e^s\}$ and is strictly increasing, too. The new parametrization coincides with that of (2.1) type a) if $s \le -1/(2\rho)$ and with that of (2.1) type c) if $s \ge -1/(4\rho)$ and $t \le -1/2$.
Suppose now the parametrizations of the S^j of type a) and c) fit together. In this case we can use continuous trial functions in the vicinity of a point P or Q of a type c) domain S^j. Therefore, let us fix a small ϵ_* and identify all the indices of $\mathscr{I}$ corresponding to one and the same point of S if the latter does not belong to an S^j of type d) or to $S^j \setminus \{\Phi^j(-\infty, 0)\}$ with S^j of type b) and if its distance to the set of edge points is less than ϵ_*. Only for points $P_{\iota_1} = P_{j_1,k_1,l_1} = P_{j_2,k_2,l_2} = P_{\iota_2}$ with larger distance or for points belonging to an S^j of type d) or to $S^j \setminus \{\Phi^j(-\infty, 0)\}$ with S^j of type b), we distinguish between P_{ι_1} and P_{ι_2} and, for a piecewise continuous function f, we let $f(P_{\iota_1})$ and $f(P_{\iota_2})$ stand for the limit of f taken from S^{j_1} and S^{j_2}, respectively. Finally, we need a set of collocation points which is uniformly distributed in the edge direction. Thus we drop the points P_ι from the set of collocation points if $\iota = (j, 1, l)$ for S^j of type a) and $\iota = (j, 1, l)$ or $\iota = (j, M_0 - 1, l)$ for type c) and if additionally the distance to the edge is less than ϵ_*. Similarly, we remove the corresponding indices from the index set $\mathscr{I}$. Note that these collocation points have only been introduced in our original setting to compensate the additional degrees of freedom due to the discontinuities of the trial functions along the common boundary of the subdomains. ∎

2.2 The trial functions

In view of the parametrization, we shall start by defining univariate splines over $[-\infty, 0]$ and $[0,1]$. Let φ stand for the cubic B-spline over the grid $\mathbb{Z} \subseteq \mathbb{R}$ such that $supp\,\varphi = [-2,2]$ and $\int_{\mathbb{R}} \varphi = 1$. For $-\infty \leq s \leq 0$ and $0 \leq t \leq 1$, we set[1] (cf.Figs.3 and 4)

$$\begin{aligned} \varphi_k^{-\infty,0}(s) &:= \begin{cases} \varphi(s/h + k - 1) & \text{if } k = 0, 1, \ldots, M_\infty - 1 \\ 1 - \sum_{k=0}^{M_\infty - 1} \varphi_k^{-\infty,0}(s) & \text{if } k = M_\infty \end{cases}, \\ \varphi_k^{0,1}(t) &:= \varphi(t/h - k + 1) \text{ if } k = 0, 1, \ldots, M_0. \end{aligned} \tag{2.4}$$

Now, for $\iota = (j,k,l) \in \mathscr{I}$ and $(s,t) \in D^j$, we introduce the bivariate tensor product spline

$$\varphi_\iota(\Phi^j(s,t)) := \begin{cases} \varphi_k^{-\infty,0}(s) \;\; \varphi_l^{-\infty,0}(t) & \text{in case a) and if } k < M_\infty \\ \varphi_k^{-\infty,0}(s) & \text{in case a) and if } k = M_\infty \\ \varphi_k^{-\infty,0}(s) \;\; \varphi_l^{0,1}(t) & \text{in case b) and if } k < M_\infty \\ \varphi_k^{-\infty,0}(s) & \text{in case b) and if } k = M_\infty \\ \varphi_k^{0,1}(s) \;\; \varphi_l^{-\infty,0}(t) & \text{in case c)} \\ \varphi_k^{0,1}(s) \;\; \varphi_l^{0,1}(t) & \text{in case d).} \end{cases} \tag{2.5}$$

Note that the support of φ_ι is contained in two or more subdomains if P_ι is an edge point or a vertex.

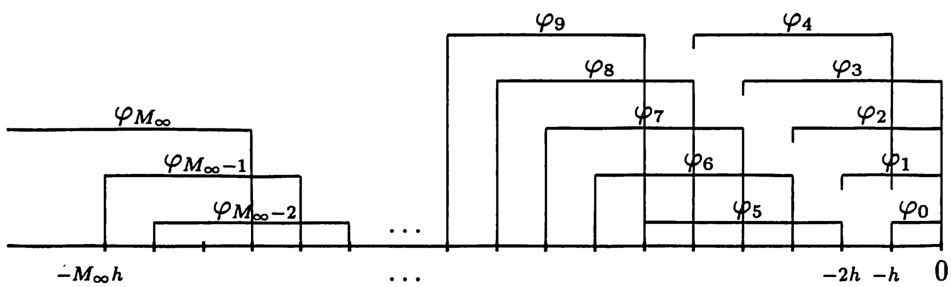

Figure 3: Supports of the functions $\varphi_k^{-\infty,0}$.

REMARK 2.2 Let us suppose as in Remark 2.1 that the parametrizations of the type a) and c) subdomains S^j fit together. Of course, we can extend the functions $s \mapsto \varphi_k^{-\infty,0}(s)$, $k = 0, \ldots, M_\infty - 1$ for $s > 0$ and $s \mapsto \varphi_k^{0,1}(s)$, $k = 0, \ldots, M_0$ for $s < 0$ as well as for $s > 1$ using the definition in (2.4). Now we define the trial functions φ_ι for ι from the index set $\mathscr{I}$ of Remark 2.1 as follows. If the distance from P_ι to the edge is greater or equal to ϵ_* or if $\iota = (j,k,l)$ with S^j of type b) or d), then φ_ι is the piecewise continuous function given by (2.5). If the distance is less than ϵ_* and if $\iota = (j,k,l)$ with S^j of type a) or c), then we define φ_ι by

[1] In practical computations we have used the B-splines with multiple nodes at the end points of the interval instead of $\varphi_k^{-\infty,0}$, $k = 0,1,2$. This choice seems to lead to better condition numbers of the stiffness matrix. The GMRES solver has been observed to be faster in this case.

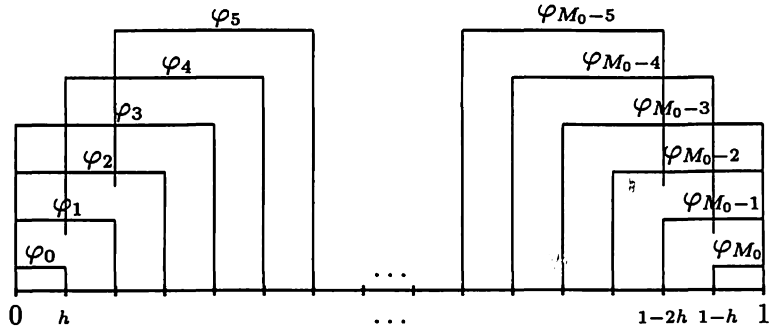

Figure 4: Supports of the functions $\varphi_k^{0,1}$.

$$\varphi_\iota(\Phi^j(s,t)) := \begin{cases} \varphi(s/h) \;\; \varphi_l^{-\infty,0}(t) & \text{in case a) and if } k=0 \\ \varphi(s/h+k-1) \;\; \varphi_l^{-\infty,0}(t) & \text{in case a) and if } 1<k<M_\infty \\ \varphi_k^{-\infty,0}(s) & \text{in case a) and if } k=M_\infty \\ \varphi(s/h-k+1) \;\; \varphi_l^{-\infty,0}(t) & \text{in case c) if } 1<k<M_0-1 \\ \varphi(s/h) \;\; \varphi_l^{-\infty,0}(t) & \text{in case c) if } k=0 \\ \varphi(s/h-M_0+2) \;\; \varphi_l^{-\infty,0}(t) & \text{in case c) if } k=M_0\ , \end{cases} \tag{2.6}$$

where now $s \in \mathbb{R}$ and Φ^j is the extension of the parametrization into the neighboring subdomain. Hence, we obtain smooth splines as trial functions near the edge. ■

2.3 The discretized collocation method

Now the (semi-discretized) collocation method for the approximate solution of the double layer integral equation $Ax = y$ with $A = I + 2W_S$ consists in seeking an approximate solution $x_N := \sum_{\iota\in\mathscr{I}} \xi_\iota\varphi_\iota$ satisfying

$$(Ax_N)(P_\iota) = y(P_\iota), \quad \iota \in \mathscr{I}. \tag{2.7}$$

By definition, this is a collocation with piecewise continuous trial functions. However, if the right-hand side y is continuous, then the properties of W_S imply that the solution x_N is continuous, too. Hence, for continuous right-hand sides, (2.7) is equivalent to the collocation with trial functions from *span* $\{\varphi_\iota : \iota \in \mathscr{I}\} \cap C(S)$ and with collocation points $\{P_\iota : \iota \in \mathscr{I}\}$, where the indices $\iota_1 = (j_1, k_1, l_1)$ and $\iota_2 = (j_2, k_2, l_2)$ corresponding to one and the same point $P_{\iota_1} = P_{\iota_2}$ of $S^{j_1} \cap S^{j_2}$ are identified and where the limit $f(P_{\iota_1})$ taken from S^{j_1} is replaced by the usual point value of the continuous function f. If $\xi := \{\xi_\iota,\ \iota \in \mathscr{I}\}$ and $\eta := \{\eta_\iota,\ \iota \in \mathscr{I}\}$, $\eta_\iota := y(P_\iota)$, then the system (2.7) is equivalent to the matrix equation $A_N\xi = \eta$, where $A_N := (a_{\iota,\kappa})_{\iota,\kappa\in\mathscr{I}}$, $a_{\iota,\kappa} := (A\varphi_\kappa)(P_\iota)$. Hence, to apply this collocation method, we have to compute the entries $a_{\iota,\kappa}$. In our discretized collocation method we shall do this by simple quadrature rules.

The first step in the discretization of the integrals of $a_{\iota,\kappa}$ is some kind of singularity subtraction technique[2]. Taking into account that the constant functions are eigenfunc-

[2]Though this technique is not necessary for the derivation of asymptotic L^2 convergence rates, the supremum norm error will not converge to zero for $N \longrightarrow \infty$ if this technique is not applied.

tions of W_S corresponding to the eigenvalue 1/2, we conclude

$$[\frac{1}{2} - d_\Omega(P_\iota)]x_N(P_\iota) + \int_S k(P_\iota, P)x_N(P)d_PS = \tag{2.8}$$
$$\frac{1}{2}x_N(P_\iota) + \int_S k(P_\iota, P)[x_N(P) - x_N(P_\iota)]d_PS,$$
$$(Ax_N)(P_\iota) = 2x_N(P_\iota) + 2\int_S k(P_\iota, P)[x_N(P) - x_N(P_\iota)]d_PS. \tag{2.9}$$

In the last integral the singular behavior of $k(P_\iota, P)$ for $P \to P_\iota$ is moderated by the factor $[x_N(P) - x_N(P_\iota)]$. Therefore, the quadrature converges faster if it is applied to the right-hand side of (2.8) than applied to (1.1). Using $\sum_{\nu\in\mathcal{J}} \varphi_\nu \equiv 1$, we arrive at

$$x_N(P) - x_N(P_\iota) = \sum_{\nu\in\mathcal{J}} [\xi_\nu - x_N(P_\iota)]\varphi_\nu(P), \tag{2.10}$$
$$a_{\iota,\kappa} = [2 - \Sigma_\iota]\varphi_\kappa(P_\iota) + 2\int_S k(P_\iota, P)\varphi_\kappa(P)d_PS, \tag{2.11}$$
$$\Sigma_\iota := \sum_{\nu\in\mathcal{J}} 2\int_S k(P_\iota, P)\varphi_\nu(P)d_PS.$$

In other words, applying the quadrature rule

$$\int_S f(P)d_PS \sim \sum_{\mu\in\mathcal{J}} f(Q_\mu)\omega_\mu \tag{2.12}$$

to (2.9) yields the approximation

$$a^+_{\iota,\kappa} := [2 - \Sigma^+_\iota]\varphi_\kappa(P_\iota) + 2\sum_{\mu\in\mathcal{J}} k(P_\iota, Q_\mu)\varphi_\kappa(Q_\mu)\omega_\mu, \tag{2.13}$$
$$\Sigma^+_\iota := \sum_{\nu\in\mathcal{J}} 2\sum_{\mu\in\mathcal{J}} k(P_\iota, Q_\mu)\varphi_\nu(Q_\mu)\omega_\mu$$

for $a_{\iota,\kappa}$. Thus our discretized collocation method consists in seeking $x_N := \sum_{\iota\in\mathcal{J}} \xi_\iota\varphi_\iota$ satisfying $A^+_N\xi = \eta$, $A^+_N := (a^+_{\iota,\kappa})_{\iota,\kappa\in\mathcal{J}}$. It remains to introduce the nodes and weights in (2.12).

2.4 The quadrature rules

Let us define the quadrature rule (2.12) for the discretization of (2.11). This rule will strongly depend on the collocation point P_ι though we suppress this dependence in the notation of (2.12). Thus we fix the collocation point $P_\iota \in S^{j_\iota}$ and start by giving the quadrature nodes and weights over a triangle S^j of case a). The underlying mesh of nodes will be of the form $\{\Phi^j(s^\&_k, s^\#_l),\ k = 0, \ldots, M^\&,\ l = 0, \ldots, M^\#\}$. The mesh $\{\Phi^j(s_k, s_l)\}$ (i.e. the choice $s^\&_k = s_k$, $s^\#_l = s_l$) given in Sect.2.1 is graded in the vicinity of the vertex P and of the edge $\overline{PQ}$. However the grading near the edge is only in the direction perpendicular to the edge. After a slight modification (cf. the subsequent introduction of $s^\#_k$, $k = M_\infty, \ldots, M^\#$), this is sufficient for the quadrature of the integral over the kernel function if the source point P_ι is close to P or far from $\overline{PQ}$. If P_ι is close to $\overline{PQ}$

and far from P, then we need a further mesh grading near P_ι in the direction parallel to the edge.

Thus we proceed as follows. We fix an integer $i_* \geq 1$, and retain the notation of s_k in (2.2). We enlarge the partition $\{s_k : k = 0, \dots, M_\infty\}$ by $i_* + 1$ additional points between s_{M_∞} and $s_{M_\infty - 1}$ such that the partition is uniform over the set $[-M_\infty h, 0]$ and that the values $\exp s_k$ for the additional nodes s_k form a uniform partition over the interval $[0, \exp(-M_\infty h)]$ [3] [4]. In other words, we introduce

$$s_k^\# := \begin{cases} s_k & \text{if } k = 0, 1, \dots, M_\infty - 1 \\ -(M_\infty - 1)h & \text{if } k = M_\infty \\ -M_\infty h & \text{if } k = M_\infty + 1 \\ -M_\infty h + \log[1 - (k - M_\infty - 1)/i_*] & \text{if } k = M_\infty + 2, \dots, M_\infty + i_* \\ -\infty & \text{if } k = M^\# := M_\infty + i_* + 1 \,. \end{cases}$$

If $S^j = S^{j_\iota}$ or if $S^j \cap S^{j_\iota} = \emptyset$ or if $S^j \cap S^{j_\iota} = \{P\}$, then we choose $M^\& := M^\#$, $s_k^\& := s_k^\#$, $k = 0, \dots, M^\&$. For the case $S^j \cap S^{j_\iota} = \overline{PQ}$, we define $s_\# := s_{k_\iota}$ for $P_\iota = \Phi^{j_\iota}(s_{k_\iota}, s_{l_\iota})$. Clearly, $\Phi^j(s_\#, 0)$ is the orthogonal projection of P_ι onto $\overline{PQ}$. Then the grid $\{\exp(s_k^\#),\ k = 0, \dots, M_\infty\}$ has a geometric mesh refinement near 0 and $\{\exp(s^\#) \pm \exp(s^\#)\exp(s_k),\ k = 0, \dots, M_\infty\} \cap [0,1]$ has this kind of refinement near $\exp(s^\#)$. We choose $\{s_k^\&,\ k = 0, \dots, M^\&\}$ such that $\{\exp(s_k^\&),\ k = 0, \dots, M^\&\}$ is the union of $\{\exp(s_k^\#),\ k = 0, \dots, M^\#\}$ and $\{\exp(s^\#) \pm \exp(s^\#)\exp(s_k),\ k = 0, \dots, M_\infty\} \cap [\exp(s^\#_{M^\# - i_*}), 1]$ and that $0 = s_0^\& > s_1^\& > \dots > s_{M^\&}^\& = -\infty$. Now, for the integration in one direction, Simpson's formula yields

$$\int_{-\infty}^0 f(e^s)e^s ds = \int_0^1 f(x)dx = \int_0^{\exp(s^\#_{M^\# - i_*})} f(x)dx + \int_{s^\#_{M^\# - i_*}}^0 f(e^s)e^s ds \tag{2.14}$$

$$\sim \sum_{k=M^\# - i_*}^{M^\# - 1} \left[e^{s_k^\#} - e^{s_{k+1}^\#}\right] \left\{\frac{1}{6} f(e^{s_k^\#}) + \frac{2}{3} f(e^{(s_k^\# + s_{k+1}^\#)/2}) + \frac{1}{6} f(e^{s_{k+1}^\#})\right\}$$

$$+ \sum_{k=0}^{M^\# - i_* - 1} [s_k^\# - s_{k+1}^\#] \left\{\frac{1}{6} f(e^{s_k^\#})e^{s_k^\#} + \frac{2}{3} f(e^{(s_k^\# + s_{k+1}^\#)/2}) e^{(s_k^\# + s_{k+1}^\#)/2} + \frac{1}{6} f(e^{s_{k+1}^\#}) e^{s_{k+1}^\#}\right\}$$

$$=: \sum_{k=0}^{2M^\# - 1} f(e^{\sigma_k})\omega_k^\# ,$$

$$\int_{-\infty}^0 f(e^s)e^s ds = \int_0^1 f(x)dx = \int_0^{\exp(s^\&_{M^\& - i_*})} f(x)dx + \int_{s^\&_{M^\& - i_*}}^0 f(e^s)e^s ds$$

[3] We remark that this partition of $[0, \exp(-M_\infty h)]$ into i_* subintervals is introduced in order to obtain a small ratio $[\exp(s_k^\#) - \exp(s_{k+1}^\#)]/\exp(s_{M_\infty - 1})$ for $k = M_\infty + 1, \dots, M^\# - 1$. This is necessary to guarantee that, for sufficiently large i_*, the discretized collocation is a small perturbation of the collocation method and thus stable. In practical computations the choice $i_* = 1$ has been sufficient. Though with this choice the discretized collocation is not a small perturbation of the collocation anymore, it should be possible to prove stability for the discretized collocation analogously to the proof presented here for the semi-discretized collocation.

[4] Note that $[-\infty, -M_\infty h]$ is just the set, where the functions $\varphi_k^{-\infty,0}$, $k = 0, \dots, M_\infty - 1$ vanish and $\varphi_{M_\infty}^{-\infty,0}$ is identically equal to one.

$$\sim \sum_{k=M^{\&}-i_\circ}^{M^{\&}-1} \left[e^{s_k^{\&}} - e^{s_{k+1}^{\&}}\right] \left\{\frac{1}{6}f(e^{s_k^{\&}}) + \frac{2}{3}f(e^{(s_k^{\&}+s_{k+1}^{\&})/2}) + \frac{1}{6}f(e^{s_{k+1}^{\&}})\right\}$$

$$+ \sum_{k=0}^{M^{\&}-i_\circ-1} [s_k^{\&} - s_{k+1}^{\&}] \left\{\frac{1}{6}f(e^{s_k^{\&}})e^{s_k^{\&}} + \frac{2}{3}f(e^{(s_k^{\&}+s_{k+1}^{\&})/2})e^{(s_k^{\&}+s_{k+1}^{\&})/2} + \frac{1}{6}f(e^{s_{k+1}^{\&}})e^{s_{k+1}^{\&}}\right\}$$

$$=: \sum_{k=0}^{2M^{\&}-1} f(e^{\tau_k})\omega_k^{\&}.$$

Finally, the nodes and weights for the quadrature over S^j are given by[5]

$$\begin{aligned}\int_{S^j} f(P)d_PS &= \int_{-\infty}^{0}\int_{-\infty}^{0} f(\Phi^j(s,t))e^{2s}e^t\left|\overrightarrow{PQ}\times\overrightarrow{QR}\right| dsdt \\ &\sim \sum_{k=0}^{2M^{\&}-1}\sum_{l=0}^{2M^{\#}-1} f(\Phi^j(\tau_k,\sigma_l))e^{\tau_k}\omega_l^{\#}\omega_k^{\&}\left|\overrightarrow{PQ}\times\overrightarrow{QR}\right|.\end{aligned} \tag{2.15}$$

Similar rules can be obtained for the subdomains of the other cases or for the parametrizations of Remark 2.1 and we omit the details. Let us mention only that, if S^j is of type b), then a rule similar to (2.15) can be formed with the partition $\{s_k^{\#}\}$ in $s-$ direction and with a uniform partition in $t-$ direction. In case c) the mesh in edge direction will be the union of a uniform mesh and a geometrically graded mesh, where the orthogonal projection of P_ι onto the edge is the accumulation point of the geometric mesh. The mesh in perpendicular direction is $\{s_k^{\#}\}$. For case d), uniform meshes are sufficient. Collecting the quadrature rules over all subdomains S^j together, gives the rule denoted by (2.12).

2.5 The modified method

Similar to the stability proof for double layer equations over curves with corners (cf.[2, 9, 15, 24]), the stability of the collocation and its discrete version hinges upon the invertibility of some "finite section" operators and their discretizations. Here the "finite section" operators are nothing else but the double layer operators $\tilde{A} := I + 2W_{\tilde{S}_P}$, where $\tilde{S}_P := S_P \setminus U_P$, S_P is the tangent cone of S at $P \in S$ and U_P is a certain neighborhood of P. Though in the one-dimensional case these finite section operators can be shown to be invertible, their invertibility in the two-dimensional case has not been proved yet. Therefore, we should introduce a modified collocation which can be shown to be stable without an assumption on the "finite sections". Let us remark, however, that this modification is of similar nature as in the one-dimensional case. In other words, the modification is recommended only if in numerical computations for the unmodified method an instability is discovered. Though the modification forces stability and shows the same asymptotic rates of convergence, we expect it to produce larger errors. Fortunately, in all our numerical tests no instability has been observed for the unmodified collocation.

[5]In practical calculations the meshsize can be a little bit larger. This means that we start with a uniform mesh over $[s^{\#}_{M^{\#}-i_\circ}, s_0]$ the mesh width of which is two times h or four times h. Following the above procedure we get a suitable quadrature rule.

In order to motivate our modification, let us remark that, roughly speaking, the matrix of the collocation (2.7) is a block Toeplitz matrix (cf. Sect.3), i.e. the restriction of a discrete convolution operator. The invertibility of the convolution operator is not hard to show. However, the invertibility of a block Toeplitz operator is a difficult problem. The idea of the modification is to multiply the Toeplitz matrix by a diagonal matrix whose entries are the values of a certain smooth cut off function. Whereas the Toeplitz matrix is a restriction of the convolution operator corresponding to the "finite section" of the double layer operator obtained by multiplying with a characteristic function, the new matrix corresponds to a "finite section" realized by multiplication with a smooth cut off function. For more details we refer to Sect.3. Now we shall define the modified method in a formal way. Let $\Upsilon : \mathbb{R} \longrightarrow \mathbb{R}^+$ denote a smooth cut off function with $\Upsilon(t) = 0$ for $t \leq 0$ and $\Upsilon(t) = 1$ for t sufficiently large. We introduce $v : S \longrightarrow \mathbb{R}^+$ by

$$v(\Phi^j(s,t)) := \begin{cases} \Upsilon(e^{M_\infty h}s) & \text{if } S^j \text{ is of type a) and b)} \\ 1 & \text{if } S^j \text{ is of type c) and d).} \end{cases}$$

Let P denote a vertex of S. For $x_N = \sum_{\iota\in\mathscr{I}} \xi_\iota\varphi_\iota$, we define $x_N^P = \sum_{\iota\in\mathscr{I}^P}(1-v)(P_\iota)\xi_\iota\varphi_\iota$, where $\mathscr{I}^P$ is the set of all $\iota = (j,k,l) \in \mathscr{I}$ such that $P \in S^j$ and $k < M_\infty$. In view of $\sum_{\iota\in\mathscr{I}} \varphi_\iota \equiv 1$, we introduce $1_N^P := \sum_{\iota\in\mathscr{I}^P}(1-v)(P_\iota)\varphi_\iota$. Then the modification of method (2.7) consists in seeking an approximate solution $x_N = \sum_{\iota\in\mathscr{I}} \xi_\iota\varphi_\iota$ satisfying

$$(Ax_N)(P_\iota) - \begin{cases} (2W_S x_N^P)(P_\iota) - x_N(P)(2W_S 1_N^P)(P_\iota) & \text{if } P_\iota \in S^k \text{ and } S^k \text{ contains the vertex } P \neq P_\iota \\ 0 & \text{else ,} \end{cases}$$
$$= y(P_\iota), \ \iota \in \mathscr{I}. \tag{2.16}$$

For the discretization of the modified collocation (2.16), we observe that the integral

$$(W_S x_N^P)(P_\iota) - x_N(P)(W_S 1_N^P)(P_\iota) = \int_S k(P_\iota,Q)[x_N^P(Q) - x_N(P)1_N^P(Q)]d_Q S$$

has already the "subtraction" form and can be computed by the quadrature rules mentioned in Sect.2.4. For the corresponding entries in the stiffness matrix of the discretized and modified collocation method, we get

$$a_{\iota,\kappa}^{++} = a_{\iota,\kappa}^{+} + \begin{cases} -2\sum_{\mu\in\mathscr{I}} k(P_\iota,Q_\mu)(1-v)(P_\kappa)\varphi_\kappa(Q_\mu)\omega_\mu & \text{if } P_\iota \in S^{k_\iota},\ P_\kappa \in S^{k_\kappa} \text{ such that } S^{k_\iota}\cap S^{k_\kappa} \text{ contains a vertex } P,\ P_\kappa \neq P \neq P_\iota \\ \sum_{\nu\in\mathscr{I}^P} 2\sum_{\mu\in\mathscr{I}} k(P_\iota,Q_\mu)(1-v)(P_\nu)\varphi_\nu(Q_\mu)\omega_\mu & \text{if } P_\iota \in S^{k_\iota} \text{ such that } S^{k_\iota} \text{ contains the vertex } P = P_\kappa \neq P_\iota \\ 0 & \text{else.} \end{cases}$$

3 LOCAL STABILITY IN THE VICINITY OF A VERTEX

3.1 Collocation over a cone

Stability and convergence of the methods (2.7) and (2.16) will follow from a localization principle. Therefore, we shall first consider the stability for a local representative, i.e. for the corresponding method applied to the double layer equation over the infinite tangent cone at the vertices. Thus let Ω denote an infinite polyhedral cone with vertex $P = 0$ and let S stand for $\partial\Omega$. In accordance with Sect.2 we shall introduce the corresponding variants of collocation methods for $Ax = y$, $A := I + 2W_S$. These methods coincide with those of Sect.2 in the vicinity of the vertex and represent natural continuations from this vicinity to the whole infinite cone S. After the definition of the methods, we shall show their stability in Sects. 3.2-3.4.

We start with a coarse partition $S = \cup_{j=1}^{J} S^j$ of S into a finite number of infinite plane sectors. Here we suppose that there exist two kinds of sectors. In case a) we assume that the corner of S^j is the vertex P of the cone and that S^j contains exactly one edge of S. In case b) we suppose that the corner of S^j is P, but S^j contains no edge point different from P. We choose points Q and R on S^j such that the half axes running from P through Q and R, respectively, form the boundary of S^j and that Q is an edge point for case a). Moreover, for neighbours S^{j_1} and S^{j_2}, we assume that the points Q or R at the joint boundary axis coincide. Similarly to (2.1) we set

$$\Phi^j : D^j := \begin{cases} [-\infty,\infty] \times [-\infty,0] & \text{in case a)} \\ [-\infty,\infty] \times [0,1] & \text{in case b)} \end{cases} \longrightarrow S^j, \tag{3.1}$$

$$\Phi^j(s,t) := \begin{cases} e^s\, \overrightarrow{PQ} + e^s e^t\, \overrightarrow{QR} & \text{in case a)} \\ e^s\, \overrightarrow{PQ} + e^s t\, \overrightarrow{QR} & \text{in case b).} \end{cases}$$

Retaining the notation of s_k, t_k, h, M_∞, M_0 of (2.2) and setting $u_k := -h(k-1)$, $k \in \mathbb{Z}$ as well as $s_k^\infty = u_k$ for $k = 1, 0, -1, \ldots$ and $s_k^\infty := s_k$ for $k = 2, 3, \ldots, M_\infty$, we define the collocation points $P_{j,k,l} := \Phi^j(s_k^\infty, s_l)$, $k = M_\infty, M_\infty - 1, \ldots,$ $l = 0, \ldots, M_\infty$ for S^j of case a) and by $P_{j,k,l} := \Phi^j(s_k^\infty, t_l)$, $k = M_\infty, M_\infty - 1, \ldots,$ $l = 0, \ldots, M_0$ for S^j of case b). By $\mathscr{I}$ we denote the set of all indices $\iota = (j,k,l)$, where we identify those indices ι_1 and ι_2 for which the point $P_{\iota_1} = P_{\iota_2}$ is on an edge. In order to define the trial functions, we retain the definitions of $\varphi^{-\infty,0}$ and $\varphi^{0,1}$ in (2.4) and set $\varphi_k^{-\infty,\infty}(s) := \varphi(s/h + k - 1)$ if $k \in \mathbb{Z}$. For $\iota = (j,k,l) \in \mathscr{I}$ and $(s,t) \in D^j$, we introduce the bivariate tensor product spline

$$\varphi_\iota(\Phi^j(s,t)) := \begin{cases} \varphi_k^{-\infty,\infty}(s) \;\; \varphi_l^{-\infty,0}(t) & \text{in case a) and if } k < M_\infty \\ \varphi_{M_\infty}^{-\infty,0}(s) & \text{in case a) and if } k = M_\infty \\ \varphi_k^{-\infty,\infty}(s) \;\; \varphi_l^{0,1}(t) & \text{in case b) and if } k < M_\infty \\ \varphi_{M_\infty}^{-\infty,0}(s) & \text{in case b) and if } k = M_\infty. \end{cases} \tag{3.2}$$

Now the corresponding collocation method (CM) over the polyhedral cone S is again given by (2.7). Moreover, defining x_N^P and 1_N^P as in Sect.2.5 but with the index set $\mathscr{I}$ introduced in this section, the modified collocation (MC) over the cone is given again by (2.16). Of course, these methods lead to an infinite system of equations and are of theoretical interest only. Beside them we shall introduce another version (IC), where we define an infinite dimensional space of trial functions even in the neighborhood of the vertex P. Namely, let us define $P^*_{j,k,l} := \Phi^j(u_k, s_l)$, $k \in \mathbb{Z}$, $l = 0, \dots, M_\infty$ for S^j of case a) and $P^*_{j,k,l} := \Phi^j(u_k, t_l)$, $k \in \mathbb{Z}$, $l = 0, \dots, M_0$ for S^j of case b). By $\mathscr{I}^*$ we denote the set of all indices $\iota = (j, k, l)$, where again those indices ι_1 and ι_2 are identified for which $P^*_{\iota_1} = P^*_{\iota_2}$ is on an edge. Now, for $\iota = (j, k, l) \in \mathscr{I}^*$ and $(s,t) \in D^j$, we define the splines by

$$\varphi^*_\iota(\Phi^j(s,t)) := \begin{cases} \varphi_k^{-\infty,\infty}(s) \ \ \varphi_l^{-\infty,0}(t) & \text{in case a)} \\ \varphi_k^{-\infty,\infty}(s) \ \ \varphi_l^{0,1}(t) & \text{in case b).} \end{cases} \tag{3.3}$$

Using these functions, we consider the collocation method IC approximating the exact solution $x = A^{-1}y$ by the solution $x^*_N = \sum_{\iota \in \mathscr{I}^*} \xi_\iota \varphi^*_\iota$ of $Ax^*_N(P^*_\iota) = y(P^*_\iota)$, $\iota \in \mathscr{I}^*$. Note that the stiffness matrix $A^*_N := (a^*_{\iota,\kappa})_{\iota,\kappa \in \mathscr{I}^*}$, $a^*_{\iota,\kappa} := A\varphi^*_\kappa(P^*_\iota)$ of this method takes the form of a block convolution matrix (cf. Sect.3.2). The matrix A_N of method CM is the truncation of A^*_N, i.e. the Toeplitz matrix corresponding to the convolution matrix A^*_N, and the matrix $A^\#_N$ of MC is another modification of A^*_N. We shall start with the stability analysis for method IC and, from this, we shall derive the stability of CM and MC.

3.2 Stability of the method IC

First let us prove the stability in the space $C(S)$. The method IC is called stable if the approximate operators A^*_N are invertible for h sufficiently small and if the sequence of inverse operators is uniformly bounded with respect to h. Since

$$\| \sum_{\iota \in \mathscr{I}^*} \xi_\iota \varphi^*_\iota \|_{L^\infty(S)} \sim \sup_{\iota \in \mathscr{I}^*} |\xi_\iota|, \tag{3.4}$$

we have to consider boundedness and stability of the approximate operators A^*_N in the supremum norm. Taking into account that $P^*_{j,k,l} = e^{-(k-1)h} P^*_{j,1,l}$, $\varphi^*_{j,k,l}(Q) = \varphi^*_{j,1,l}(e^{(k-1)h}Q)$ and $W_S x(t \cdot Q) = [W_S x(t \cdot)](Q)$, we conclude $a^*_{(j_\iota,k_\iota,l_\iota),(j_\kappa,k_\kappa,l_\kappa)} = a^*_{(j_\iota,1,l_\iota),(j_\kappa,k_\kappa-k_\iota+1,l_\kappa)}$ and

$$A^*_N = \left(A^*_{N,k_\iota-k_\kappa}\right)_{k_\iota,k_\kappa \in \mathbb{Z}}, \quad A^*_{N,k} := \left(a^*_{(j_\iota,1,l_\iota),(j_\kappa,k+1,l_\kappa)}\right)_{(j_\iota,l_\iota),(j_\kappa,l_\kappa) \in \Lambda}, \tag{3.5}$$

where Λ stands for the index set of all (j,l) with $j = 1, \dots, J$ and with $l = 0, \dots, M_\infty$ for S^j of type a) and $l = 0, \dots, M_0$ for S^j of type b). Of course, we identify those indices $\lambda_1 = (j_1, l_1)$ and $\lambda_2 = (j_2, l_2)$ which correspond to one and the same edge point $P^*_{(j_1,1,l_1)} = P^*_{(j_2,1,l_2)}$. From (3.5) we conclude that A^*_N is a discrete convolution operator. Let us denote by $A(\omega)$ its symbol function $[-\pi, \pi] \ni \omega \mapsto A(\omega) := \sum_{k \in \mathbb{Z}} e^{i\omega k} A^*_{N,k} \in \mathscr{L}(l^\infty(\Lambda))$. It is a well known fact that the invertibility of $A(\omega)$ is necessary for the invertibility of A^*_N. Moreover, together with some other properties of the symbol the invertibility of $A(\omega)$ will imply the invertibility of A^*_N and the uniform boundedness of $(A^*_N)^{-1}$. Therefore, we start by showing the invertibility of $A(\omega)$. We conclude

$$\begin{aligned} A(\omega) &= \left(A \left(\sum_{k \in \mathbb{Z}} e^{i\omega k} \varphi^*_{(j_\kappa,k+1,l_\kappa)} \right) (P^*_{(j_\iota,1,l_\iota)}) \right)_{(j_\iota,l_\iota),(j_\kappa,l_\kappa) \in \Lambda} \\ &= \left(A(g^\omega_N \otimes \varphi_{(j_\kappa,l_\kappa)})(P_{(j_\iota,l_\iota)}) \right)_{(j_\iota,l_\iota),(j_\kappa,l_\kappa) \in \Lambda}, \end{aligned} \tag{3.6}$$

where the following notation is used. By $P_{(j_\iota,l_\iota)}$ we denote the points $P_{(j_\iota,l_\iota)} := P^*_{(j_\iota,1,l_\iota)}$ of the curve $\Gamma := \{\Phi^j(0,t) : \; j = 1,\dots,J, \; t \in [-\infty,0]$ for S^j of type a) and $t \in [0,1]$ for S^j of type b)$\} \subseteq S$. The functions $\varphi_{(j_\kappa,l_\kappa)}$ over Γ are defined by

$$\varphi_{(j_\kappa,l_\kappa)}(\Phi^{j_\kappa}(0,t)) := \begin{cases} \varphi_{l_\kappa}^{-\infty,0}(t) & \text{if } S^{j_\kappa} \text{ is of type a)} \\ \varphi_{l_\kappa}^{0,1}(t) & \text{if } S^{j_\kappa} \text{ is of type b),} \end{cases} \tag{3.7}$$

and the function g_N^ω over $[0,\infty)$ is given by $g_N^\omega(s) := \sum_{k\in\mathbb{Z}} e^{i\omega k}\varphi_{k+1}^{-\infty,\infty}(s)$. Finally, we let $g_N^\omega \otimes \varphi_{(j,l)}$ stand for the tensor product function $S \ni \Phi^j(s,t) \mapsto g_N^\omega(s)\cdot\varphi_{(j,l)}(\Phi^j(0,t))$. In other words, the symbol $A(\omega)$ is nothing else than the stiffness matrix of the collocation with node points $\{P_\lambda, \lambda \in \Lambda\}$ and trial functions $\{\varphi_\lambda, \lambda \in \Lambda\}$ applied to the operator $A_\Gamma^\omega \in \mathscr{L}(C(\Gamma))$,

$$A_\Gamma^\omega f(P) := A(g_N^\omega \otimes f)(P). \tag{3.8}$$

Introducing the integral operators K_N^ω and K^ω over Γ by

$$\begin{aligned} K_N^\omega f(P) &:= \int_\Gamma k_N^\omega(P,Q)f(Q)d_Q\Gamma \qquad (3.9)\\ K^\omega f(P) &:= \int_\Gamma k^\omega(P,Q)f(Q)d_Q\Gamma \\ k_N^\omega(P,\Phi^j(0,t)) &:= 2\int_{\mathbb{R}} k(P,\Phi^j(s,t))\frac{g_N^\omega(s)}{g_N^\omega(0)}e^{2s}|\,\overrightarrow{PQ}\,|ds, \qquad (3.10)\\ k^\omega(P,\Phi^j(0,t)) &:= 2\int_{\mathbb{R}} k(P,\Phi^j(s,t))e^{i\omega s}e^{2s}|\,\overrightarrow{PQ}\,|ds, \end{aligned}$$

we can prove the following properties of the operator A_Γ^ω.

LEMMA 3.1 *i) The operator A_Γ^ω takes the form $A_\Gamma^\omega = g_N^\omega(0)\{I+K_N^\omega\}$ and is invertible for any $-\pi \le \omega \le \pi$ if only $N \sim j_{lev}^2 2^{2j_{lev}}$ is sufficiently large. The norms $\|(A_N^\omega)^{-1}\|$ are even uniformly bounded with respect to N.*

ii) The kernel k_N^ω satisfies $|k_N^\omega| \le |k_N^0|$ and, if we restrict the operator K_N^0 to the neighborhood Γ_1 of a corner of Γ, then $\|K_N^0|_{\Gamma_1}\| < 1$.

iii) For any $\delta > 0$ fixed and $N \to \infty$, there holds

$$\sup_{\substack{|P-Q|>\delta \\ \omega > 2^{-j_{lev}/2}}} |k_N^\omega(P,Q)| \to 0, \qquad \sup_{\substack{P,Q\in\Gamma \\ \omega \le 2^{-j_{lev}/2}}} |k_N^\omega(P,Q) - k^{\omega/h}(P,Q)| \to 0.$$

iv) The function g_N^ω satisfies $g_N^\omega(0) > 0$, $|g_N^\omega(s)/g_N^\omega(0)| \le 1$ and, for $N \longrightarrow \infty$, $\sup_{s\in\mathbb{R},\omega\le 2^{-j_{lev}/2}} |g_N^\omega(s)/g_N^\omega(0) - e^{is\omega/h}| \longrightarrow 0$.

v) The operator function $[-\pi,\pi] \ni \omega \mapsto A_\Gamma^\omega$ is continuously differentiable with respect to ω.

PROOF: iv) If $F\varphi$ is the Fourier transform of the B-spline φ, then we get

$$\begin{aligned}\varphi(\sigma) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(F\varphi)(t)e^{-it\sigma}dt,\\ \varphi(\sigma+k) &= \frac{1}{\sqrt{2\pi}}\sum_{l\in\mathbb{Z}}\int_{-\pi}^{\pi}(F\varphi)(t+2\pi l)e^{-itk}e^{-it\sigma}e^{-i2\pi l\sigma}dt.\end{aligned}$$

Replacing σ by s/h in the last equality and substituting this into the definition of g_N^ω, we arrive at

$$\begin{aligned}g_N^\omega(s) &= \sum_{k\in\mathbb{Z}}\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\left\{\sum_{l\in\mathbb{Z}}(F\varphi)(t+2\pi l)e^{-i2\pi ls/h}e^{-its/h}\right\}e^{-itk}dte^{i\omega k}\\ &= \sqrt{2\pi}\sum_{l\in\mathbb{Z}}(F\varphi)(2\pi l+\omega)e^{-i2\pi ls/h}e^{-i\omega s/h} \qquad (3.11)\end{aligned}$$

for $-h/2 \le s \le h/2$. Note that in the last step we have used the formula for the inversion of the Fourier series expansion. The choice $s = 0$ in (3.11) and $F\varphi \ge 0$ yield $g_N^\omega(0) > 0$ as well as

$$\left|\frac{g_N^\omega(s)}{g_N^\omega(0)}\right| = \left|\frac{\sum_{l\in\mathbb{Z}}(F\varphi)(2\pi l+\omega)e^{-i2\pi ls/h}e^{-i\omega s/h}}{\sum_{l\in\mathbb{Z}}(F\varphi)(2\pi l+\omega)}\right| \le 1.$$

The last assertion of iv) follows from $\sum_{k\in\mathbb{Z}}\varphi(s+k) \equiv 1$ and the compactness of the support of φ.

iii) The second assertion of iii) is a consequence of iv) and of the boundedness of $\int_\Gamma |k_N^0(P,Q)|d_Q\Gamma$ following from $A_\Gamma^0 \in \mathscr{L}(C(\Gamma))$. In order to prove the first assertion, it will be sufficient to show $\int_{\mathbb{R}} f(s)g_N^\omega(s)ds \longrightarrow 0$ for smooth functions f, $\omega > 2^{-j_{lev}/2}$, and $N \longrightarrow \infty$. Since smooth functions can be approximated by piecewise constant ones, we may suppose that f is the characteristic function of the interval $[A,B] \subseteq \mathbb{R}$. If k is the greatest integer less than $-A/h+2$ and l the smallest integer greater than $-B/h-2$, then we get

$$\begin{aligned}\int_A^B g_N^\omega(s)ds &= \sum_{j=k-4}^{k} e^{i\omega j}\int_A^B \varphi(s/h+k)ds + \sum_{j=k+1}^{l-1} e^{i\omega j}\int_{\mathbb{R}}\varphi(s/h+k)ds\\ &\quad + \sum_{j=l}^{l+4} e^{i\omega j}\int_A^B \varphi(s/h+k)ds = O(h) + h\sum_{j=k+1}^{l-1} e^{i\omega j}\\ \left|\int_A^B g_N^\omega(s)ds\right| &\le O(h) + h\frac{1}{|1-e^{i\omega}|} = O(2^{-j_{lev}/2}).\end{aligned}$$

ii) The first assertion follows easily from iv) and $g_N^0 \equiv 1$. The second one is a consequence of the well-known estimate $\|2W_{S_1}\| < 1$ valid for the boundary S_1 of a wedge.

i) The representation of A_Γ^ω is obvious from the definition of A_Γ^ω and K_N^ω. If $\omega > 2^{-j_{lev}/2}$, then the first assertion of iii) and ii) imply $\|K_N^\omega\| < 1$ and A_Γ^ω is invertible. On the other hand, for $\omega \le 2^{-j_{lev}/2}$, the second assertion of iii) leads to $\|K_N^\omega - K^{\omega/h}\| \longrightarrow 0$. Thus, in view of $g_N^\omega(0) \longrightarrow 1$, we conclude $\|A_\Gamma^\omega - (I+K^{\omega/h})\| \longrightarrow 0$. Since $I+K^{\omega/h}$ is the Mellin symbol of $A \in \mathscr{L}(C(S))$ and since the latter is invertible (cf.[32]), A_Γ^ω is invertible, too.

v) Since the absolute value of the function $\partial_\omega g_N^\omega(s) = i\sum_{k\in\mathbb{Z}} e^{i\omega k} k\varphi(s/h+k)$ grows only slightly faster than $|g_N^\omega(s)|$ as $|s| \longrightarrow \infty$, the boundedness of $\partial_\omega A_\Gamma^\omega$ is not hard to show. ■

LEMMA 3.2 *i) The collocation operator $A(\omega) \in \mathscr{L}(l^\infty(\Lambda))$ is invertible for any $-\pi \leq \omega \leq \pi$ if only N is large enough. The norms $\|A(\omega)^{-1}\|$ are uniformly bounded with respect to N and to ω.*

ii) The operator function $[-\pi,\pi] \ni \omega \mapsto A(\omega)$ is continuously differentiable with respect to ω.

PROOF: i) Let P_N denote the interpolation projection of $C(\Gamma)$ onto the linear hull of $\{\varphi_\lambda,\ \lambda\in\Lambda\}$ interpolating at the points $\{P_\lambda,\ \lambda\in\Lambda\}$. Then $P_N f = \sum_{\lambda\in\Lambda} f(P_\lambda)\psi_\lambda$, where $\psi_\lambda(P_\kappa) = \delta_{\lambda,\kappa}$. This projection exists and is uniformly bounded for large N. To see this it is sufficient to show the uniform boundedness of the matrices $(\varphi_k^{-\infty,0}(s_l))_{l,k=0,\dots,M_\infty}$ and $(\varphi_k^{0,1}(t_l))_{l,k=0,\dots,M_0}$. In view of the theorems on finite section operators (cf. e.g.Chapt.4 of [28]), it remains to prove the invertibility of $(\varphi_k^{-\infty,\infty}(v_l))_{l,k=0,\dots} \in \mathscr{L}(l^\infty)$, where $v_l := s_l$ for $l=0,1$ and $v_l = u_l$ for $l=2,3,\dots$. Note that $(\varphi_k^{-\infty,0}(s_l))_{l,k=0,\dots,M_\infty}$ is a finite section of $(\varphi_k^{-\infty,\infty}(v_l))_{l,k=0,\dots}$. However, the last matrix is the perturbation of an invertible Toeplitz matrix by a finite rank operator. Hence, it will be sufficient to show that the null space of the matrix in l^∞ is trivial. Now it is not hard to prove that any cubic spline over $[v_0,v_2]$ vanishing at v_0,v_1,v_2 admits exactly one continuation as a cubic spline vanishing at the other nodes v_k, $k=3,\dots$. This function of the null space of the matrix is bounded if and only if the function is identically zero. Thus the null space in l^∞ is trivial and P_N is uniformly bounded.

For the interpolation basis $\{\psi_\lambda\}$, we get $\psi_\lambda = \sum_{\kappa\in\Lambda} b_{\lambda,\kappa}\varphi_\kappa$ with $b_{\lambda,\lambda}=1$ if P_λ is a corner point of Γ. Moreover, $b_{(j_1,l_1),(j_2,l_2)} = 0$ if $j_1 \neq j_2$ and $P_{(j_1,l_1)}$ and $P_{(j_2,l_2)}$ do not coincide with one and the same corner point of Γ or if $j_1 = j_2$ and $P_{(j_2,l_2)}$ is a corner point but $P_{(j_1,l_1)}$ is not. From Lemma 2 of [21], we deduce $|b_{(j,l_1),(j,l_2)}| \leq ce^{-\epsilon|l_1-l_2|}$, where $c>0$ and $\epsilon>0$ are independent of j, l_1, l_2, and N. The collocation operator $A(\omega) := P_N A_\Gamma^\omega|_{im\,P_N}$ is given in (3.6) by its matrix representation with respect to the bases $\{\varphi_\lambda,\ \lambda\in\Lambda\}$ and $\{\psi_\lambda,\ \lambda\in\Lambda\}$. To prove stability, we write A_Γ^ω in the form $A_\Gamma^\omega = g_N^\omega(0)\{I + T_N^\omega + R_\Gamma^\omega\}$, where $T_N^\omega = \sum_{m=1}^{M_C} \chi_m I K_N^\omega \chi_m I$ and χ_m denotes a cut off function which is identically equal to one in a vicinity of the m-th corner of Γ and identically zero in the vicinities of the other corners. From Lemma 3.1 we conclude that R_Γ^ω is a small perturbation of the collectively compact family of operators $\sum_{m=1}^{M_C}[1-\chi_m] I K^{\omega/h}\chi_m I + K^{\omega/h}[1-\sum_{m=1}^{M_C}\chi_m]I$ if $\omega \leq 2^{-j_{lev}/2}$. For $\omega > 2^{-j_{lev}/2}$, the operator R_Γ^ω is small. Furthermore, from Lemma 3.1 ii) we conclude that $(I+T_N^\omega)$ is invertible. Hence, by standard arguments (cf.e.g. Chapt.1 of [28]), it is sufficient to prove stability for $P_N(I+T_N^\omega)|_{im\,P_N}$ instead of $P_N A_\Gamma^\omega|_{im\,P_N}$.

Now we define $P_N^1 f := \sum f(P_\lambda)\psi_\lambda$ and $P_N^2 := P_N - P_N^1$, where the summation runs over those indices $\lambda\in\Lambda$ for which P_λ is an edge point. We observe that, corresponding to the splitting $im\,P_N = im\,P_N^1 \oplus im\,P_N^2$, the operator $P_N(I+T_N^\omega)|_{im\,P_N}$ has the matrix representation

$$P_N(I+T_N^\omega)|_{im\,P_N} = \begin{pmatrix} I|_{im\,P_N^1} & 0 \\ P_N^2 T_N^\omega|_{im\,P_N^1} & P_N^2(I+T_N^\omega)|_{im\,P_N^2} \end{pmatrix}. \tag{3.12}$$

Consequently, it suffices to prove $P_N^2(I+T_N^\omega)|_{im\,P_N^2}$ to be stable. Let us introduce $1_N := P_N^2 1$ and let $1_N I$ stand for the operator of multiplication by 1_N. Next we shall show

$\|(1_N I - P_N^2)T_N^\omega P_N^2\| \longrightarrow 0$ for $N \longrightarrow \infty$. If this is done, then the stability of $P_N^2(I + T_N^\omega)|_{im\,P_N^2}$ follows from

$$\begin{aligned} &\left\{P_N^2(I + [1_N I]T_N^\omega)^{-1}|_{im\,P_N^2}\right\}\left\{P_N^2(I + T_N^\omega)|_{im\,P_N^2}\right\} = \\ &\qquad I|_{im\,P_N^2} + P_N^2(I + [1_N I]T_N^\omega)^{-1}(P_N^2 - 1_N I)T_N^\omega|_{im\,P_N^2}, \\ &\|P_N^2(I + [1_N I]T_N^\omega)^{-1}(P_N^2 - 1_N I)T_N^\omega|_{im\,P_N^2}\| \not\longrightarrow 0. \end{aligned}$$

Here we have used the existence of $(I + [1_N I]T_N^\omega)^{-1}$ which can be shown as follows. We observe $1_N = 1 - \sum \psi_\lambda$, where the summation runs over all $\lambda \in \Lambda$ such that P_λ is a corner point of Γ. Especially, 1_N is uniformly bounded. Moreover, from $b_{\lambda,\lambda} = 1$ and the exponential decay of $b_{\lambda,\kappa}$ we conclude that, for any $\epsilon > 0$, there exists a positive integer L such that $\{P \in \Gamma \cap S^j : 1_N(P) > 1+\epsilon\}$ is contained in $\{\Phi^j(0,t) : u_{M_\infty+2} < t < u_{M_\infty-L}\}$. The diameter of the last set divided by its distance to the corner point $\Phi^j(0,-\infty)$ can be estimated by $O(\,(\exp u_{M_\infty-L} - \exp u_{M_\infty+2})/\exp u_{M_\infty+2}\,)$ and tends to 0 for $N \longrightarrow \infty$. Thus the angle under which the set $\{P \in \Gamma \cap S^j : 1_N(P) > 1+\epsilon\}$ is seen from any point of Γ tends to zero. Using this fact and the interpretation of the integral of the double layer kernel $k^0 = k_N^0$ as an angle, we obtain $\|T_N^\omega[1_N I]\| \leq 1 - \delta$ for a small $\delta > 0$ and sufficiently large N. Hence, the spectral radius $\rho(\,T_N^\omega[1_N I]\,) = \rho(\,[1_N I]T_N^\omega\,)$ is less than one and $I + [1_N I]T_N^\omega$ is invertible.

In order to prove $\|(1_N I - P_N^2)T_N^\omega P_N^2\| \longrightarrow 0$, we take a point $R = \Phi^j(0,u_R)$ in the support of 1_N and suppose without loss of generality that S^j is of type a). From the exponential decay of the coefficients $b_{\kappa,\lambda}$, we infer that, for any $\epsilon > 0$, there exists an integer $L > 0$ independent of R, N and an integer l_R depending on R, N such that $\sum_{|l-l_R|>L} |\psi_{(j,l)}(R)| < \epsilon$. The definition of 1_N implies

$$\begin{aligned} (1_N I - P_N^2)T_N^\omega P_N^2 f(R) &= 1_N(R)T_N^\omega P_N^2 f(R) - \sum_{\kappa\in\Lambda:\,\psi_\kappa\in im\,P_N^2} T_N^\omega P_N^2 f(P_\kappa)\psi_\kappa(R) \\ &= \sum_{\kappa\in\Lambda:\,\psi_\kappa\in im\,P_N^2} \left[T_N^\omega P_N^2 f(R) - T_N^\omega P_N^2 f(P_\kappa)\right]\psi_\kappa(R) \\ &= \Sigma_1 + \Sigma_2, \end{aligned} \tag{3.13}$$

where Σ_1 is the sum over all $\kappa = (j,l)$ with $|l - l_R| \leq L$ and Σ_2 that over the rest. Obviously, $|\Sigma_2| \leq 2\,\|T_N^\omega\|\,\epsilon\,\|P_N^2 f\|$. For κ from the summation in Σ_1, we get $|R - P_\kappa|/dist(\overline{RP_\kappa}, \Phi^j(0,\infty)) \longrightarrow 0$ for $N \longrightarrow \infty$. Setting $x := (g_N^\omega/g_N^\omega(0)) \otimes P_N^2 f$, we get

$$\begin{aligned} K_N^\omega P_N^2 f(R) - K_N^\omega P_N^2 f(P_\kappa) &= \frac{1}{2\pi}\int_S \left\{\frac{n_Q R}{|R-Q|^3} - \frac{n_Q P_\kappa}{|P_\kappa - Q|^3}\right\} x(Q)d_Q S, \\ |K_N^\omega P_N^2 f(R) - K_N^\omega P_N^2 f(P_\kappa)| &\leq \frac{1}{2\pi}\int_S c\frac{|R-P_\kappa|}{dist(\overline{RP_\kappa}, \Phi^j(0,\infty))}\left|\frac{n_Q R}{|R-Q|^3}\right| x(Q)d_Q S, \\ &\leq c\frac{|R-P_\kappa|}{dist(\overline{RP_\kappa}, \Phi^j(0,\infty))}\|x\|_{L^\infty(S)} \\ &\leq c\frac{|R-P_\kappa|}{dist(\overline{RP_\kappa}, \Phi^j(0,\infty))}\|P_N^2 f\|_{L^\infty(\Gamma)}, \end{aligned} \tag{3.14}$$

where c is a certain constant. From the last estimate it is not hard to deduce that $|T_N^\omega P_N^2 f(R) - T_N^\omega P_N^2 f(P_\kappa)|/\|P_N^2 f\| \longrightarrow 0$ for $N \longrightarrow \infty$, and $\|(1_N I - P_N^2)T_N^\omega P_N^2\| \longrightarrow 0$

follows from (3.13).

ii) This assertion follows immediately from the corresponding assertion in Lemma 3.1 v). ∎

We have shown that the block convolution matrix A_N^* has a symbol which is continuously differentiable and invertible[6] on $[-\pi,\pi]$ for N large enough. Moreover, the norms of the symbol, of its inverse, and of its derivative are uniformly bounded with respect to N and ω. Hence, the inverse symbol function belongs to the Wiener class and the Wiener norm is uniformly bounded with respect to N. All these facts together imply that A_N^* is invertible and the inverse is uniformly bounded. In other words, the stability of the method IC with respect to the supremum norm is proved.

The proof for the L^2 stability runs analogously. Instead of (3.4) one starts from an estimate including weighted l^2 norms. Thus the stability of the collocation matrix IC in these weighted norms is equivalent to the l^2 stability of the matrix multiplied by weights. This new matrix is a convolution matrix, too. A corresponding version of Lemma 3.1 holds, where g_N^ω is to be defined including the additional weights from the norm equivalence and the operator $I + K^{\omega/h}$ is to be replaced by the L^2 Mellin symbol (cf. [17]). For the boundedness and invertibility of the convolution matrix in the l^2 case, it is sufficient to show the boundedness of the symbol $A(\omega)$ and of its inverse $A(\omega)^{-1}$. This can be shown with the arguments of the proof to Lemma 3.2. We note only that the interpolation projection P_N is not bounded anymore and, therefore, the uniform boundedness of the collocation matrix must be proved additionally. Estimates similar to (3.14) prove that $P_N^2(I + T_N^\omega)|_{im\,P_N^2}$ is a small perturbation of the Galerkin operator.

3.3 Stability of the method CM

Let us consider the method CM (cf. Sect.3.1) for the operator $A = I + 2W_S \in \mathscr{L}(L^2(S))$ and the corresponding matrix A_N acting in the space of vectors ξ endowed with the norm

$$\|\{\xi_\iota\}\|_* := \|\sum_{\iota\in\mathscr{I}} \xi_\iota\varphi_\iota\|_{L^2(S)}.$$

For the formulation of the stability theorem, we need a notation. We set $S' := \{\Phi^j(s,t) : j = 1,\dots,J,\ (s,t) \in D^j,\ s \geq 0\}$ and let A' stand for the double layer operator $A' = I + 2W_{S'} \in \mathscr{L}(L^2(S'))$.

THEOREM 3.3 *The method CM applied to the double layer operator $A = I + 2W_S$ over the boundary S of an infinite polyhedral cone is stable if and only if the "finite section" operator A' is invertible.*

PROOF: a) The proof of the sufficiency is analogous to that of Theorem 2.1 in [31]. Let us denote by ι_0 the index from $\mathscr{I}$ such that $P_{\iota_0} = 0$ is the vertex of S. We set $\mathscr{I}_0 = \mathscr{I}\setminus\{\iota_0\}$ and define the interpolation projection P_N [7] mapping the set of piecewise continuous functions over S onto the linear hull of $\{\varphi_\iota : \iota \in \mathscr{I}_0\}$ by $P_N f(P_\iota) = f(P_\iota)$, $\iota \in \mathscr{I}_0$. The existence of this projection follows by the arguments at the beginning of the proof to Lemma 3.2. We introduce $A_N' := (a_{\kappa,\iota})_{\kappa,\iota\in\mathscr{I}_0}$ as well as $A_N'' := (a_{\kappa,\iota_0})_{\kappa\in\mathscr{I}_0}$ to get

[6] Note that the symbol, its derivative, and its inverse have coinciding limits at $-\pi$ and π.

[7] This projection is different from that of Sect.3.2.

$$A_N = \begin{pmatrix} A'_N & A''_N \\ 0 & 2[1 - d_\Omega(0)] \end{pmatrix}. \tag{3.15}$$

Furthermore, by $A^N \in \mathscr{L}(L^2(S^N))$ we denote the double layer operator $A^N := I + 2W_{S^N}$, where S^N stands for the truncated boundary $S^N := \cup_{\iota \in \mathscr{I}_0} supp\, \varphi_\iota$. Obviously, $R \in S^N$ is equivalent to $e^{M_\infty h} R \in S'$. Hence, by homogeneity arguments the invertibility of A' is equivalent to that of A^N, and $\|(A^N)^{-1}\| = \|(A')^{-1}\|$. Now the image $im\, P_N$ can be identified with the space of functions over the set of interpolation points $\{P_\iota\}_{\iota \in \mathscr{I}_0}$. Consequently, A'_N maps $im\, P_N$ into $im\, P_N$. By $W'_N \in \mathscr{L}(im\, P_N)$ we denote the operator which can be obtained analogously to the definition of A'_N if we start with W_S instead of A, i. e., $W'_N := 1/2(A'_N - I)$. For a function χ on S^N, we set $\chi_N := P_N \chi I|_{im\, P_N}$. Finally, by χ^δ we denote the characteristic function of the set of points $R \in S$ whose distance to the set of edge points is less than $|R|\delta$. Now, in view of Lemma 2.2 and of the proof to Theorem 2.1 in [31], all we have to show is that the following assertions are valid.

i) The operators A''_N and A'_N are uniformly bounded with respect to N.

ii) Let $\delta > 0, \epsilon > 0$. Then, for N large enough, we get $\|(1 - \chi^\delta) I [A^N - A'_N]|_{im\, P_N}\| < \epsilon$.

iii) There exist $N_0 > 0, \delta_0 > 0$ such that , for $N \geq N_0$ and $\delta \leq \delta_0$, the operator $[I + 2W'_N(\chi^\delta)_N] \in \mathscr{L}(im\, P_N)$ is invertible and that the norms of the inverse operators are uniformly bounded.

b) Let us show i) and start by proving that $A'_N := P_N A|_{im\, P_N}$ is bounded. In the last section we have sketched the proof of the fact that the matrix A^*_N is a block convolution operator with a bounded symbol function. Consequently, A^*_N is bounded. Since A'_N is just the restriction of A^*_N to a subspace defined by a truncation of the index set, A'_N is bounded, too. In order to prove the boundedness of $A''_N = P_N A|_{span\{\varphi_{\iota_0}\}}$, we observe that

$$\varphi_{\iota_0}(\Phi^j(s,t)) = \begin{cases} \sum_{k \geq M_\infty, l=0,\dots,M_\infty} \varphi_k^{-\infty,\infty}(s) \varphi_l^{-\infty,0}(t) & \text{for } S^j \text{ of case a)} \\ \sum_{k \geq M_\infty, l=0,\dots,M_0} \varphi_k^{-\infty,\infty}(s) \varphi_l^{0,1}(t) & \text{for } S^j \text{ of case b).} \end{cases}$$

Hence, the matrix of $P_N A|_{span\{\varphi_{\iota_0}\}}$ takes the form $A^*_N b_N$ with $b_N := (\vartheta_\iota)_{\iota \in \mathscr{I}^*}$, $\vartheta_\iota := 0$ if $\iota = (j,k,l)$ with $k < M_\infty$ and $\vartheta_\iota := 1$ if $\iota = (j,k,l)$ with $k \geq M_\infty$. Since the function $\sum_{\iota \in \mathscr{I}^*} \vartheta_\iota \varphi^*_\iota$ is in $L^2(S)$ and A^*_N is bounded, the operator $P_N A|_{span\{\varphi_{\iota_0}\}}$ is bounded, too.
c) Let us prove assertion ii). Thus we have to prove that $\|(1-\chi^\delta) I [A^N - P_N A]|_{im\, P_N}\| < \epsilon$. We write $(1-\chi^\delta) I [A^N - P_N A]|_{im\, P_N} = 2(Te_1 + Te_2)$, where $Te_1 := (1-\chi^\delta) I [(1_N I) W_S - P_N W_S]|_{im\, P_N}$, $1_N := P_N 1$, and $Te_2 := (1-\chi^\delta)[1 - 1_N] I\, W_{S'}|_{im\, P_N}$. Let $\chi := \chi(N)$, $\tilde{\chi} := \tilde{\chi}(N)$, and χ^* stand for the characteristic functions of the sets $\cup_{j=1,\dots,J}\{\Phi^j(s,t) : (s,t) \in D^j,\ u_{M_\infty - l} \leq s \leq u_{M_\infty + 2}\}$, $\cup_{j=1,\dots,J}\{\Phi^j(s,t) : (s,t) \in D^j,\ u_{-l} \leq s \leq u_2\}$, and $\cup_{j=1,\dots,J}\{\Phi^j(s,t) : (s,t) \in D^j,\ 1/2 \leq s \leq 2\}$, respectively. Then we observe that $1 - 1_N$ is a bounded function. Since the function ψ_κ of the interpolation basis defined by $\psi_\kappa(P_\iota) = \delta_{\iota,\kappa}$, $\iota, \kappa \in \mathscr{I}_0$ satisfies an exponential decay estimate (compare the proof of Lemma 3.2), we conclude that, for a sufficiently large but fixed l, $\chi(1 - 1_N) - (1 - 1_N)$ is small over S'. Hence, for $\|Te_2\| \leq \epsilon/4$, it is sufficient to show that $(1 - \chi^\delta)\chi I\, W_S$ has a small norm. By homogeneity arguments, we get $\|(1-\chi^\delta)\chi I\, W_S\| = \|(1-\chi^\delta)\tilde{\chi} I\, W_S\|$ and $\|(1-\chi^\delta)\tilde{\chi} I\, W_S\| = \|\tilde{\chi}(1-\chi^\delta)\chi^* I\, W_S\|$. The last norm, however, tends to zero as $N \to \infty$

since $(1-\chi^\delta)\chi^* I\, W_S$ is compact and the operator of multiplication by $\tilde{\chi} := \tilde{\chi}(N)$ tends strongly to zero.
Let W^+ denote the integral operator over S with the kernel function $|k(Q,R)|$ which is the absolute value of the kernel of W_S and let $\psi_\iota \in im\, P_N$ be defined by $\psi_\iota(P_\kappa) = \delta_{\iota,\kappa}$. In order to estimate Te_1, we consider an arbitrary $R = \Phi^j(s_R, t_R) \in supp(1-\chi)\cap S^j$. From the exponential decay of the functions ψ_ι we infer that there exist an integer $L > 0$ depending on ε and integers k_R, l_R depending on R, N, ε such that $\sum_{|l-l_R|>L,\,|k-k_R|>L} |\psi_{(j,k,l)}(R)| < \varepsilon[16\|W_S\|]^{-1}$. We conclude

$$|[(1_N I)W_S - P_N W_S]P_N f(R)| = \left|\sum_{\iota\in\mathscr{I}_0}[W_S(P_N f)(R) - W_S(P_N f)(P_\iota)]\psi_\iota(R)\right|$$

$$\le \frac{\varepsilon}{8}\|P_N f\| + \sum_{\substack{|l-l_R|\le L\\ |k-k_R|\le L}} \frac{W_S(P_N f)(R) - W_S(P_N f)(P_{(j,k,l)})}{W^+(|P_N f|)(R)} W^+(|P_N f|)(R)\psi_{(j,k,l)}(R),$$

$$\le \frac{\varepsilon}{8}\|P_N f\| + c \sup_{\substack{|l-l_R|\le L\\ |k-k_R|\le L}} \left|\frac{W_S(P_N f)(R) - W_S(P_N f)(P_{(j,k,l)})}{W^+(|P_N f|)(R)}\right| \|W^+\|\|P_N f\| \tag{3.16}$$

Hence, it remains to estimate the supremum. We conclude

$$[W_S(P_N f)(R) - W_S(P_N f)(P_{(j,k,l)})] = \int_S [k(R,Q) - k(P_{(j,k,l)},Q)]P_N f(Q) d_Q S,$$

$$|W_S(P_N f)(R) - W_S(P_N f)(P_{(j,k,l)})| \le \sup\left|\frac{k(R,Q) - k(P_{(j,k,l)},Q)}{k(R,Q)}\right| \int_S |k(R,Q)|\,|P_N f(Q)| d_Q S,$$

$$\sup_{\substack{|l-l_R|\le L\\ |k-k_R|\le L}} \left|\frac{W_S(P_N f)(R) - W_S(P_N f)(P_{(j,k,l)})}{W^+(|P_N f|)(R)}\right| \le \sup\left|\frac{k(R,Q) - k(P_{(j,k,l)},Q)}{k(R,Q)}\right|, \tag{3.17}$$

where the last sup is taken over k, l with $|k - k_R| \le L$, $|l - l_R| \le L$ and over $Q \in S$ such that Q is not on the same face of S as S^j. Now the estimate $\|Te_1\| \le \epsilon/4$ follows from (3.16) and the fact (cf. proof of Lemma 2.2 ii) in [31]) that the supremum on the right-hand side of (3.17) tends to zero if $N \to \infty$.
d) Let us consider assertion iii). By χ_e^δ we denote the characteristic function of the set of points $R \in S$ such that the distance of R to a given edge e of S is less than $|R|\delta$. Furthermore, let X_2 stand for the linear span of all φ_ι, $\iota \in \mathscr{I}_0$ such that $\chi_e^\delta(P_\iota) = 1$ but P_ι does not belong to the edge e. Following the proof of Lemma 2.2 iii) in [31], we only have to show $\|P_N\chi_e^\delta I\, W_N'|_{X_2}\| < 1/2$. However, $P_N\chi_e^\delta I\, W_N'|_{X_2} = P_N\chi_e^\delta I\, W_S|_{X_2}$, and it remains to prove $\|P_N\chi_e^\delta I\, W_S|_{X_2}\| < 1/2$. From the proof in Sect.3.2 we infer that the last operator is the restriction of a convolution operator to a subspace defined by the truncation of the index set. The corresponding convolution operator has a symbol with norm less than one half. Since the restriction operator is of norm one and the Fourier transform is unitary in l^2, we conclude $\|P_N\chi_e^\delta I\, W_S|_{X_2}\| < 1/2$. The proof of the sufficiency is finished.

e) Now let us assume that the method CM is stable. Then the sequence of matrices $A'_N = P_N A|_{im\,P_N} \in \mathscr{L}(im\,P_N)$ is stable, too. Let us introduce Tf for $f \in \mathscr{L}(L^2(S))$ by $Tf(R) := f(\exp(u_{M_\infty+1})R)$. We observe $TA = AT$ and that the interpolation projections $P_N^T := T^{-1}P_NT \in \mathscr{L}(L^2(S'))$ tend strongly to the identity. From the stability of $T^{-1}A'_NT = P_N^T A'|_{im\,P_N^T}$ and the strong limit $T^{-1}A'_NTP_N^T \longrightarrow A'$, we conclude that the null space of A' is trivial. Since it is well known that A' is Fredholm and that $Ind\,A' = 0$, A' is invertible. ∎

3.4 Stability of the method MC

THEOREM 3.4 *For any polyhedral cone S, there exists a cut off function Υ such that the modified method (2.16) over the cone S is stable with respect to the supremum norm.*
PROOF: The collocation matrix A_N^* has been shown to be stable in Sect.3.2. Moreover, the matrix $A_N^\#$ splits into four block matrices analogously to A_N in (3.15) and the stability of $A_N^\#$ will follow from that of the left upper block like the stability of A_N from that of A'_N. This left upper block matrix takes the form $\mathscr{P} + \mathscr{P}Cv_N \in \mathscr{L}(im\,\mathscr{P})$, where $C := A_N^* - I$, $\mathscr{P} = (p_{\iota,\kappa})_{\iota,\kappa\in\mathscr{I}^*}$, $v_N = (v_{\iota,\kappa})_{\iota,\kappa\in\mathscr{I}^*}$, and

$$p_{\iota,\kappa} := \begin{cases} 1 & \text{if } \iota=\kappa \text{ and } \iota\in\mathscr{I}_0 \\ 0 & \text{else} \end{cases}, \quad v_{\iota,\kappa} := \begin{cases} v(P_\iota) & \text{if } \iota=\kappa \text{ and } \iota\in\mathscr{I}_0 \\ 0 & \text{else} \end{cases},$$

From $\mathscr{P}v_N = v_N$, we infer that $\mathscr{P} + \mathscr{P}Cv_N \in \mathscr{L}(im\,\mathscr{P})$ is invertible if and only if $I + Cv_N \in \mathscr{L}(l^\infty(\mathscr{I}^*))$ is invertible. Hence, we only have to prove the next lemma. ∎

LEMMA 3.5 *i) The sequence of matrices $B_N := I + Cv_N \in \mathscr{L}(l^\infty)$ is stable if the operator $G := I + 2W_Sv_0I \in \mathscr{L}(C(S))$ is invertible, where $v_0 : S \longrightarrow \mathbb{R}^+$ is given by $v_0(\Phi^j(s,t)) := \Upsilon(s)$.*

ii) Take a fixed cut off function Υ^* (cf.Sect.2.5) and, for $\rho > 0$, define $\Upsilon(t) := \Upsilon^*(\rho t)$ as well as $v_0(\Phi^j(s,t)) := \Upsilon(s)$. Then there is a small ρ such that G is invertible.

PROOF: i) The proof will follow from the localization principle of [19] and standard arguments (cf. e.g. [28]). Let us introduce some notation. By P_N^* we denote the projection onto the span of $\{\varphi_\iota^*,\ \iota\in\mathscr{I}^*\}$ interpolating at $\{P_\iota^*,\ \iota\in\mathscr{I}^*\}$. Then we get $P_N^*f = \sum_{\iota\in\mathscr{I}^*} f(P_\iota)\psi_\iota^*$ and $P_N^* \in \mathscr{L}(C^*(S))$ tends strongly to the identity if $C^*(S)$ is the space of all $f \in C(S)$ vanishing at zero and infinity. We introduce Tf for $f \in C(S)$ by $Tf(P) := f(\exp(u_{M_\infty})R)$ and observe $TP_N^* = P_N^*T$ as well as $W_ST = TW_S$. Though A_N^* is the stiffness matrix of the collocation operator $P_N^*A|_{im\,P_N^*}$ with respect to the bases $\{\varphi_\iota^*\}, \{\psi_\iota^*\}$, in the present proof we shall identify each matrix $E_N \in \mathscr{L}(l^\infty(\mathscr{I}^*))$ (e.g.$E_N = A_N^*$, B_N, $\mathscr{P}$, v_N) with that operator of $\mathscr{L}(im\,P_N^*)$ whose matrix representation with respect to the bases $\{T\varphi_\iota\}, \{T\psi_\iota\}$ is E_N. With this identification, we observe $\|v_N - P_N^*v_0I|_{im\,P_N^*}\| \longrightarrow 0$ and get $v_NP_N^* \longrightarrow v_0I$ and $B_NP_N^* \longrightarrow G$. For any function χ over S, we define the operator χ_N by $P_N^*\chi I|_{im\,P_N^*}$. Now suppose $1 = \sum_{i=1}^{i+} \chi^i$ is a continuous partition of unity over S such that v_0 is nearly equal to the constant value v^i on the support of χ^i. Then the first candidate for the inverse of B_N is $D_N^1 := \sum_{i=1}^{i+} \chi_N^i(B_N^i)^{-1}$, where $B_N^i := I + Cv^i$. The existence and uniform boundedness of $B_N^i = P_N^*(I + 2v^iW_S)|_{im\,P_N^*} \in \mathscr{L}(l^\infty)$ follows analogously to the proof for A_N^* in Sect.3.2. We get

$$\begin{aligned} B_N D_N^1 &= \sum (B_N - B_N^i)\chi_N^i (B_N^i)^{-1} + \sum [B_N^i \chi_N^i - \chi_N^i B_N^i](B_N^i)^{-1} + I \\ &= I + \Sigma_1 + \Sigma_2, \end{aligned}$$

where $\Sigma_1 := \sum(B_N - B_N^i)\chi_N^i(B_N^i)^{-1}$ is small by the smallness of $v_0 - v^i$ over the support of χ^i. Moreover, $[B_N^i\chi_N^i - \chi_N^i B_N^i] = P_N^* 2v^i[W_S\chi^i I - \chi^i I W_S]|_{im\, P_N^*} - P_N^* 2v^i W_S(P_N^* - I)\chi^i I|_{im\, P_N^*}$ and $\Sigma_3 := \sum_{i=1}^{i+} P_N^* 2v^i W_S[(I-P_N^*)\chi^i I P_N^*](B_N^i)^{-1}$ tends to zero in the operator norm. The operator $C^i := 2v^i[W_S\chi^i I - \chi^i I W_S]$ is compact. Setting $D_N^2 := D_N^1 - \sum_{i=1}^{i+} P_N^* G^{-1} C^i P_N^* (B_N^i)^{-1}$, we obtain

$$\begin{aligned} B_N D_N^2 &= I + \Sigma_1 + \Sigma_3 + \Sigma_4, \\ \Sigma_4 &:= \sum \{P_N^* - B_N P_N^* G^{-1}\} C^i P_N^* (B_N^i)^{-1}. \end{aligned}$$

Since C^i is compact and $\{P_N^* - B_N P_N^* G^{-1}\}$ tends to 0 strongly, we get $\|\Sigma_4\| \longrightarrow 0$ for $N \longrightarrow \infty$. In other words, B_N has the uniformly bounded right inverse $D_N^2\{I + \Sigma_1 + \Sigma_3 + \Sigma_4\}^{-1}$. A similar right inverse exists for $I + \lambda C v_N$ with $0 \le \lambda \le 1$. Thus $I + \lambda C v_N$ is invertible for small λ and invertible from the right for $0 \le \lambda \le 1$. This implies the invertibility of $B_N = I + \lambda C v_N, \lambda = 1$.
ii) The operator $v_0 I W_S - W_S v_0 I$ is an integral operator with kernel $2k(R,Q)[v_0(R) - v_0(Q)]$. Using the interpretation of the integral of k as a solid angle and the smallness of the derivative of v_0 for small ρ, we conclude that the commutator $v_0 I W_S - W_S v_0 I$ is small for small ρ. This fact and $\|(2W_S)^k\| < 1 - \epsilon$ for a large integer k and a small positive ϵ (cf. [32]) imply $\|(2W_S v_0 I)^k\| < 1 - \epsilon/2$. Hence, $G := I + 2W_S v_0 I \in \mathscr{L}(C(S))$ is invertible. ■

4 GLOBAL STABILITY

4.1 Local representatives for the global stability

In the last section we have considered the vertex $P = 0$, the tangent cone S of the polyhedron at this vertex and the corresponding double layer operator over the cone. For this operator, we have shown that the corresponding collocation is stable. The latter collocation operator is a local representative at P for the collocation operator of the method introduced over the polyhedron in Sect.2. Similarly, we have to analyze the stability of local representatives at all the other boundary points of the polyhedron. However, if we consider a point in the interior of a face, then the tangent cone is a plane, the double layer operator $I + 2W_S$ is the identity, and the stability of the collocation method is trivial. Thus it remains to check the case of an edge point. We start with the definition of the corresponding collocation method.

Let S denote the union of two half planes $S = \cup_{j=1}^2 S^j$ with a common edge. We assume that the edge $S^1 \cap S^2$ contains the points $P = 0$ and Q, and that $R = R^j$, $j = 1,2$ is located in the interior of the face S^j. Similarly to (2.1) we set

$$\Phi^j : D^j := [-\infty,\infty] \times [-\infty,\infty] \longrightarrow S^j, \quad \Phi^j(s,t) := s\ \overrightarrow{PQ} + e^t\ \overrightarrow{QR}\ . \tag{4.1}$$

Retaining the definition of h and s_k^∞ from (2.2) and Sect.3.1 and setting $s_k^e := hk$, $k \in \mathbb{Z}$, we introduce the collocation points $P_{j,k,l} := \Phi^j(s_k^e, s_l^\infty)$, $j = 1,2$, $k \in \mathbb{Z}$, $l = M_\infty, M_\infty -$

$1,\ldots$. By $\mathscr{I}$ we denote the set of all indices $\iota = (j,k,l)$, where we identify those indices ι_1 and ι_2 for which the point $P_{\iota_1} = P_{\iota_2}$ is on the edge. In order to define trial functions, we retain the definitions of $\varphi_k^{-\infty,0}$, $\varphi_k^{-\infty,\infty}$ from Sects.2.2 and 3.1 and set $\varphi_k^e(s) := \varphi(s/h-k)$ if $k \in \mathbb{Z}$. For $\iota = (j,k,l) \in \mathscr{I}$ and $(s,t) \in D^j$, we introduce the bivariate tensor product spline

$$\varphi_\iota(\Phi^j(s,t)) := \begin{cases} \varphi_k^e(s) \quad \varphi_l^{-\infty,\infty}(t) & \text{if } l < M_\infty \\ \varphi_k^e(s) \quad \varphi_{M_\infty}^{-\infty,0}(t) & \text{if } l = M_\infty \,. \end{cases} \tag{4.2}$$

Now the corresponding collocation method (CM) over the boundary S of the wedge is given by (2.7). Since the modification in Sect.2.5 is done only in the vicinity of vertices, the local representative for the modified collocation coincides with the just introduced representative of the unmodified method.

THEOREM 4.1 *The method (2.7) applied to the double layer operator over the boundary S of a wedge is stable with respect to the supremum and L^2- norms.*

PROOF: The proof is completely analogous to that in Sect.3.2. ■

Unfortunately, the collocation operator of the last theorem is a local representative of the collocation over a polyhedron only at those edge points, where the parametrization is smooth in edge direction. Let us consider a corner point P or Q of S^j of type a) or type c) and suppose that the partition $S = \cup S^j$ is chosen in such a manner that the first derivatives in edge direction of the two different parametrizations coincide at this point P or Q. For this case, the localized collocation is defined by the following trial functions and collocation points: We take the basis functions φ_ι which are contained in $\{\Phi^j(s,t) : s \leq 0,\, j = 1,2\}$ or in $\{\Phi^j(s,t) : s \geq 0,\, j = 1,2\}$. The other functions φ_ι are replaced by their restrictions to $\{\Phi^j(s,t) : s \leq 0,\, j = 1,2\}$ and $\{\Phi^j(s,t) : s \geq 0,\, j = 1,2\}$, respectively. Additionally to the already defined collocation nodes, we introduce the collocation points $\Phi^j(\pm h/2, s_l^\infty)$, $j = 1,2$, $l = M_\infty, M_\infty - 1,\ldots$. It is not hard to derive that the matrix corresponding to this new collocation is a compact perturbation of the collocation in Theorem 4.1 and, therefore, it is a Fredholm operator with index zero depending on the meshsize h. To show its invertibility, however, seems to be a hard problem. This is just the reason why we have introduced the new parametrizations and the corresponding collocation nodes from Remark 2.1 together with the splines from Remark 2.2. Namely, for the collocation over the polyhedron based on Remarks 2.1 and 2.2, the localized collocation at any edge point is the one mentioned in Theorem 4.1. So we suppose in our further consideration of this section that our collocation is defined as in Remarks 2.1 and 2.2.

4.2 The localization principle

Let us retain the notation of Sect.2 and consider the collocation (2.7). For the boundary S of a bounded polyhedron and a vertex $P \in S$, let S_P stand for the tangent cone of S at P and Φ_P^j, $j = 1,\ldots,J_P$ for the corresponding parametrizations $\Phi_P^j : D_P^j \longrightarrow S_P^j \subseteq S_P$. These parametrizations are defined as in (3.1), and we suppose that the parametrizations of S coincide with the parametrizations of S_P on $S \cap S_P$. Analogously to the definition of S' for the cone S in Sect.3.3, let us introduce the truncated cone S'_P for S_P. Define the function $v_{P,0} : S_P \longrightarrow \mathbb{R}^+$ by $v_{P,0}(\Phi_P^j(s,t)) := \Upsilon(s)$. Moreover, let us set $A'_P :=$

$I + 2W_{S'_P} \in \mathscr{L}(L^2(S'_P))$ and $G_P := I + 2W_{S_P} v_{P,0} I \in \mathscr{L}(C(S_P))$. From the stability of the local representatives in the Sects.3 and 4.1 we conclude

THEOREM 4.2 *i) The method (2.7) (defined as in Remarks 2.1 and 2.2) applied to the double layer operator over the boundary S of a polyhedron is L^2-stable if, for any vertex P of S, the "finite section" operator A'_P is invertible. Moreover, the method (2.7) remains stable if we replace the entries $a_{\iota,\kappa}$ by their discretizations $a^+_{\iota,\kappa}$. In other words, the discretized collocation is L^2-stable, too.*

ii) Consider the method (2.16) (defined as in Remarks 2.1 and 2.2) applied to the double layer operator over the boundary S of a polyhedron and suppose that the cut off function Υ is chosen such that the operators G_P are invertible for any vertex P of S. Then (2.16) is stable in the supremum norm. Moreover, the method (2.16) remains stable if we replace the entries by their discretizations $a^{++}_{\iota,\kappa}$.

PROOF: a) We start by proving that the discretized collocation operator $A^+_N = (a^+_{\kappa,\iota})_{\kappa,\iota \in \mathscr{I}}$ is a small perturbations of the collocation operator $A_N = (a_{\kappa,\iota})_{\kappa,\iota\in\mathscr{I}}$. If this is done, then in the following part of the proof we only have to deal with the stability for the collocation without discretization step. The discretization step for the modified method can be treated analogously.

For the sake of simplicity, we consider the quadratures corresponding to the parametrizations of (2.1) instead of those of Remark 2.1. Thus let us estimate the error of the quadrature $a^+_{\kappa,\iota}$ applied to the integral in $a_{\kappa,\iota}$. For definiteness consider S^j of type a). Set $Set := \{\Phi^j(s,t) : s^\&_k \le s \le s^\&_{k-1},\ s^\#_l \le t \le s^\#_{l-1}\}$ and, again for definiteness, suppose $k \le M^\& - i_*$, $l \le M^\# - i_*$. Denoting the quadrature nodes and Simpson weights of (2.15) over Set by $Q_i = \Phi^j(\sigma^i, \tau^i)$ and θ_i, respectively, we arrive at

$$\int_{Set} k(P_\kappa, P)\varphi_\iota(P) d_P S - \sum_i k(P_\kappa, Q_i)\varphi_\iota(Q_i)\theta_i = \tag{4.3}$$
$$\int_{s^\&_k}^{s^\&_{k-1}} \int_{s^\#_l}^{s^\#_{l-1}} \left[k(P_\kappa,P)|D\Phi^j(s,t)| - k(P_\kappa,P')|D\Phi^j(s',t')|\right] \varphi_\iota(P) ds dt -$$
$$\sum_i \left[k(P_\kappa,Q_i)|D\Phi^j(\sigma^i,\tau^i)| - k(P_\kappa,P')|D\Phi^j(s',t')|\right] \varphi_\iota(Q_i)\theta_i,$$

where we have set $P = \Phi^j(s,t)$ and where $P' = \Phi^j(s',t')$ is a fixed point of Set. Note that we have used that our quadrature is exact for the polynomial φ_ι over Set. The double integral on the right-hand side of (4.3) can be estimated by

$$\left|\int_{s^\&_k}^{s^\&_{k-1}} \int_{s^\#_l}^{s^\#_{l-1}} \left[k(P_\kappa,P)|D\Phi^j(s,t)| - k(P_\kappa,P')|D\Phi^j(s',t')|\right] \varphi_\iota(P) ds dt\right| \le$$
$$\int_{Set} |k(P_\kappa,P)|\,|\varphi_\iota(P)| d_P S \left\{ \sup_{P\in Set} \left|\frac{k(P_\kappa,P') - k(P_\kappa,P)}{k(P_\kappa,P)}\right| + \right.$$
$$\left. \sup_{P\in Set}\left|\frac{k(P_\kappa,P')}{k(P_\kappa,P)}\right| \sup_{P\in Set}\left|\frac{|D\Phi^j(s,t)| - |D\Phi^j(s',t')|}{|D\Phi^j(s,t)|}\right| \right\}.$$

For the second term, we conclude

$$\left|\sum_i \left[k(P_\kappa,Q_i)|D\Phi^j(\sigma^i,\tau^i)| - k(P_\kappa,P')|D\Phi^j(s',t')|\right]\varphi_\iota(Q_i)\theta_i\right| \leq$$

$$c\sup_i|\varphi_\iota(Q_i)|\,|k(P_\kappa,P')|\,|Set|\left\{\sup_i\left|\frac{k(P_\kappa,Q_i)-k(P_\kappa,P')}{k(P_\kappa,P')}\right| + \right.$$

$$\left.\sup_i\left|\frac{k(P_\kappa,Q_i)}{k(P_\kappa,P')}\right|\sup_i\left|\frac{|D\Phi^j(\sigma^i,\tau^i)|-|D\Phi^j(s',t')|}{|D\Phi^j(s',t')|}\right|\right\},$$

$$\sup_i|\varphi_\iota(Q_i)|\,|k(P_\kappa,P')|\,|Set| \leq c\sup_{P\in Set}\left|\frac{k(P_\kappa,P')}{k(P_\kappa,P)}\right|\int_{Set}|k(P_\kappa,P)|\,|\varphi_\iota(P)|d_PS.$$

Repeating the estimation of Sect.2 in [31], we get

$$\left|\frac{k(P_\kappa,P)-k(P_\kappa,P')}{k(P_\kappa,P)}\right| \leq c\frac{|P'-P|}{|P_\kappa-P|}, \tag{4.4}$$

where the last ratio is small by the special choice of the partition in Section 2.4. Namely, the introduction of the additional points $\{\exp(s^\#)\pm\exp(s^\#)\exp(s_k),\ k=0,\ldots,M_\infty\}\cap[\exp(s^\#_{M^\#-i_*}),1]$ in the definition of the points $\{\exp(s^\&_k),\ k=0,\ldots,M^\&\}$ guarantees that the diameter of Set is small in comparison to the distance $|P_\kappa-P|$. From these and analogous arguments we derive

$$\sup_{P\in Set}\left|\frac{k(P_\kappa,P')-k(P_\kappa,P)}{k(P_\kappa,P')}\right| \leq c\varepsilon,\quad \sup_{P\in Set}\left|\frac{k(P_\kappa,P')}{k(P_\kappa,P)}\right| \leq c, \tag{4.5}$$

$$\sup_{P\in Set}\left|\frac{k(P_\kappa,P')-k(P_\kappa,P)}{k(P_\kappa,P)}\right| \leq c\varepsilon,\quad \sup_{P\in Set}\left|\frac{k(P_\kappa,P)}{k(P_\kappa,P')}\right| \leq c \tag{4.6}$$

for arbitrary $\varepsilon>0$ if only N is sufficiently large. Using this and the estimate

$$\sup_{P\in Set}\left|\frac{|D\Phi^j(s,t)|-|D\Phi^j(s',t')|}{|D\Phi^j(s,t)|}\right| \leq c\left|\frac{e^{2s}e^t-e^{2s'}e^{t'}}{e^{2s}e^t}\right| \leq ch,$$

we get that

$$\left|\int_{Set}k(P_\kappa,P)\varphi_\iota(P)d_PS-\sum_i k(P_\kappa,Q_i)\varphi_\iota(Q_i)\theta_i\right| \leq c\varepsilon\int_{Set}|k(P_\kappa,P)|\,|\varphi_\iota(P)|d_PS, \tag{4.7}$$

and $|a_{\kappa,\iota}-a^+_{\kappa,\iota}|$ is less than $c\varepsilon\, b_{\kappa,\iota}$ with

$$b_{\kappa,\iota} := \int_S|k(P_\kappa,P)|\,|\varphi_\iota(P)|d_PS.$$

Note that $b_{\kappa,\iota}$ is the entry of a collocation matrix corresponding to the integral operator W^+ with the kernel $|k(Q,P)|$ and using the ansatz functions $|\varphi_\iota|$ instead of φ_ι. The boundedness of the latter collocation operator follows analogously to the boundedness of the original collocation operator defined for A. Hence, the matrix $(|a_{\kappa,\iota}-a^+_{\kappa,\iota}|)_{\kappa,\iota}$ has a norm less than $c\varepsilon$ and the proof of the fact that the discretized collocation is a small perturbation of the semi-discretized collocation is finished.

b) Now we let A_N stand for the collocation operator with matrix $(a_{\iota,\kappa})_{\iota,\kappa}$ considered in the subspace $span\{\varphi_\iota,\ \iota\in\mathscr{I}\}\subseteq L^2(S)$ or for the modified collocation operator considered in the subspace $span\{\varphi_\iota,\ \iota \in \mathscr{I}\} \subseteq L^\infty(S)$. For any point U of S, we denote the tangent cone by S_U and consider the "corresponding" collocation for the double layer equation over S_U. We denote the collocation points by P_ι^U, $\iota \in \mathscr{I}^U$, the basis functions by φ_ι^U. More exactly, the "corresponding" collocation is defined as follows: If U is a vertex, then we consider the method introduced in Section 3.1, where the parametrizations onto S_U are the natural continuations of those defined in the vicinity of $U \in S$. For U in the interior of a face of S, we take any set of points and any interpolation basis over the plane S_U that coincides in a neighborhood of U with $\{P_\iota\} \subseteq S$ and the spline basis $\{\varphi_\iota\}$, respectively. Using these splines as ansatz functions and these points as collocation knots we get the corresponding method over S_U. Finally let us consider an edge point U. For definiteness, we suppose that $U = \Phi^j(t_U, 0)$ with S^j of type a) and that $\Phi^j(s_{k_U}, 0)$ is the collocation point on the edge nearest to U. We define the parametrization Φ_U^j onto the face S_U^j of the wedge S_U just as in Sect.4.1, where the points $P = P_U$ and $Q = Q_U$ in formula (4.1) are taken such that $\partial_s \Phi^j(s_{k_U}, 0) = \partial_s \Phi_U^j(0,0)$ and $\Phi^j(s_{k_U}, t) = \Phi_U^j(0,t)$. With respect to this mapping, we consider the method of Section 4.1 to be the corresponding collocation method over S_U.

In any case, let $A_{U,N}$ stand for the matrix of the collocation method over S_U, $P_{U,N}$ for the projection $P_{U,N} : C(S_U) \longrightarrow span\{\varphi_\iota^U : \iota \in \mathscr{I}_{\mathscr{U}}\}$ interpolating at $\{P_\iota^U : \iota \in \mathscr{I}_{\mathscr{U}}\}$ and $\chi_{U,N}$ for the operator $P_{U,N}\chi_U I|_{im P_{U,N}}$ if χ_U is a function over S_U. Analogously we define P_N and χ_N over S. Moreover, for the L^2 analysis, we denote the orthogonal projection onto the spaces $im\, P_{U,N} \subseteq L^2(S_U)$ and $im\, P_N \subseteq L^2(S)$ by $Q_{U,N}$ and Q_N, respectively. For the analysis in the supremum norm, we simply set $Q_{U,N} := P_{U,N}$ and $Q_N := P_N$. Now our theorem follows from the proof of Lemma 3.2 in [31] and the well-known Gohberg-Krupnik localization principle (cf. [19]). We only have to verify the following assumptions.

i) The operator $A_N Q_N$ tends strongly to A.

ii) If $U \in S$ and χ_U is a smooth function over S_U with finite limit at ∞, then there is a compact operator T_U such that

$$[A_{U,N}, \chi_{U,N}] := A_{U,N}\chi_{U,N} - \chi_{U,N}A_{U,N} = Q_{U,N}T_U|_{im\, P_{U,N}} + o(1) \quad (N \to \infty).$$

iii) If χ is a smooth function over S, then there is a compact operator T such that

$$[A_N, \chi_N] = Q_N T|_{im\, P_N} + o(1) \quad (N \to \infty).$$

iv) For any $V \in S$ and any $\varepsilon > 0$, there is a neighborhood $N_V \subseteq S \cap S_V$ of V such that

$$\|\chi_N A_N \chi_N - \chi_N A_{V,N} \chi_N\| \leq \varepsilon \tag{4.8}$$

if χ is a smooth function with $|\chi| \leq 1$ and support in N_V.

v) For any $V \in S$, the method with the approximate operator $A_{V,N} \in \mathscr{L}(im\, P_{V,N})$ is stable.

c) Assumption v) follows from the previous two sections. So let us start by proving iv). If V is a vertex, then we choose $N_V \subseteq \cup_{j:V \in S^j}\{\Phi^j(s,t) : -\infty \leq s \leq -1/2,\ -\infty \leq t \leq 0\}$ and the norm in (4.8) is even zero. For a point V in the interior of a face, we choose

N_V on this face to be sufficiently small, and (4.8) holds again with $\varepsilon = 0$. Thus we may suppose that V is an edge point. In a neighborhood of V we define $\Phi : S_V^j \hookrightarrow S^j$ by $\Phi(\Phi_V^j(s,t)) := \Phi^j(s_{k_V} + s, t)$ to get $\Phi(P_{(j,k,l)}^V) = P_{(j,k_V+k,l)}$ and $\varphi_{(j,k_V+k,l)} \circ \Phi = \varphi_{(j,k,l)}^V$. We identify $\varphi_{(j,k,l)}^V$ with $\varphi_{(j,k_V+k,l)}$, and the term $\chi_N A_{V,N} \chi_N$ in (4.8) is to be understood via this identification. Thus we have to estimate $a_{\kappa^V,\iota^V}^V - a_{\kappa,\iota}$, where $a_{\kappa^V,\iota^V}^V := (I + 2W_{S_V})\varphi_{\iota^V}^V(P_{\kappa^V}^V)$, $a_{\kappa,\iota} := A\varphi_\iota(P_\kappa)$, and the indices $\kappa^V, \iota^V \in \mathscr{I}^V$, $\kappa, \iota \in \mathscr{I}$ are connected by $(j',k',l') = \kappa^V$, $(j', k_V + k', l') = \kappa$ and $(j,k,l) = \iota^V$, $(j, k_V + k, l) = \iota$. For simplicity, let us suppose $M_\infty > l$, $M_\infty > l'$ and $k' > k$. Now we conclude

$$a_{\kappa,\iota} = 2\int k(P_\kappa, P)\varphi_\iota(P) d_P S = 2\int k(\Phi(P_{\kappa^V}^V), \Phi(P))\varphi_\iota \circ \Phi(P)|D\Phi(P)| d_P S,$$

$$a_{\kappa,\iota} - a_{\kappa^V,\iota^V}^V = 2\int \left[k(\Phi(P_{\kappa^V}^V), \Phi(P))|D\Phi(P)| - k(P_{\kappa^V}^V, P)\right] \varphi_\iota \circ \Phi(P) d_P S \qquad (4.9)$$

Taking into account that the derivative of Φ at V is the identity, we obtain $|\,|D\Phi(P)| - 1| \leq \varepsilon$ and $|\Phi(P) - P| \leq \varepsilon |P - V|$ for a small prescribed ε and for P from a small neighborhood of V. Moreover, if e stands for the edge $\{\Phi^j(s,-\infty), -\infty < s < 0\} \cap \{\Phi_V^j(s,-\infty), s \in \mathbb{R}\}$, then it is not hard to verify that $|[dist(\Phi(P),e) - dist(P,e)]/dist(P,e)| < \varepsilon$ if P is taken from a small neighborhood of V. From these estimates we conclude (cf. Section 2 of [31]) that

$$\sup_{P \in supp\, \varphi_{\iota^V}^V} \left| \frac{k(\Phi(P_{\kappa^V}^V), \Phi(P)) - k(P_{\kappa^V}^V, P)}{k(P_{\kappa^V}^V, P)} \right| \leq c\varepsilon$$

for any prescribed $\varepsilon > 0$ and a sufficiently small neighborhood of V. Hence, the term on the right-hand side of (4.9) is less than $c\varepsilon \int |k(P_{\kappa^V}^V, P)|\, |\varphi_{\iota^V}^V(P)| d_P S$, i.e., less than the entry of a bounded collocation matrix (for the boundedness cf. the arguments in Sect. 3) multiplied by $c\varepsilon$. Thus $A_{V,N}$ is a small perturbation of A_N in the neighborhood of V, and iv) follows.

d) Let us show i). For definiteness, let us restrict our considerations to the unmodified method. We first prove that the approximate operators $A_N := P_N A|_{im\, P_N}$ are uniformly bounded with respect to N. Note that the restriction of W_S acting between $S_1 \subseteq S$ and $S_2 \subseteq S$ is a compact and smoothing operator if the distance between S_1 and S_2 is positive. Thus it is sufficient to show that, for any $V \in S$, there is a small neighborhood N_V of V such that A_N restricted to N_V is bounded. This, however, follows from the boundedness of the local representatives and the assertion iv) which was just proved. Knowing the uniform boundedness of A_N, it remains to show $A_N Q_N f \to Af$ for any f from a dense subset. Hence, for the L^2 convergence, we may suppose that f is smooth and vanishes in a neighborhood N_f of the edges. Again, W_S restricted to $S \setminus N_f$ is smoothing, and $A_N Q_N f \to Af$ follows from the well-known results for the collocation applied to compact integral operators with smooth kernel. For the L^∞ convergence over a dense subset, we refer to the estimations in the proof of Theorem 5.1.

e) Let us prove iii) for the unmodified method considered in the space $L^2(S)$ and $C(S)$. We obtain

$$\begin{aligned}[A_N, \chi_N] &= P_N A P_N \chi I|_{im\, P_N} - P_N \chi I P_N A|_{im\, P_N} \qquad (4.10)\\ &= Q_N T|_{im\, P_N} + \{P_N - Q_N\} T|_{im\, P_N} + P_N A\{P_N - I\}\chi I|_{im\, P_N},\end{aligned}$$

where $T := [A, \chi] = A\chi I - \chi I A$ is bounded from $C(S)$ or $L^2(S)$ to $C(S)$ and compact as an operator acting in $C(S)$ or $L^2(S)$. (Note that the kernel $l(P,Q)$ of $A\chi I - \chi IA$ is bounded by $c\delta|P-Q|^{-2}$, where δ denotes the distance from P to the plane Pl containing the face F of S such that $Q \in F$. Hence, $\int_F |l(P,Q)|^2 d_Q S$ is less than $c\delta^2 \int_F |P-Q|^{-4} \le c\delta^2 \int_{\delta \le r} r^{-3} dr \le c$.) Consequently, $\|\{P_N - Q_N\}T|_{im\, P_N}\| \to 0$. Namely, for a given $\varepsilon > 0$, let χ_1 denote the characteristic function of a small neighborhood of the set of all edge points such that $\int_S \chi_1 \le \varepsilon$. Then we arrive at

$$\begin{aligned}\|\{P_N - Q_N\}TP_Nf\|_{L^2}^2 &\le \int |\chi_1\{P_N - Q_N\}TP_Nf|^2 + \int |[1-\chi_1]\{P_N - Q_N\}TP_Nf|^2 \\ &\le \varepsilon \|\{P_N - Q_N\}TP_Nf\|_{L^\infty}^2 + \|[1-\chi_1]I\{P_N - Q_N\}TP_Nf\|_{L^2}^2 \\ &\le c\varepsilon \|P_Nf\|_{L^2}^2 + \|[1-\chi_1]I\{P_N - Q_N\}TP_Nf\|_{L^2}^2,\end{aligned}$$

where the second term on the right-hand side is bounded by $\|[1-\chi_1]I\{P_N - Q_N\}T|_{im\, P_N}\|^2 \times \|P_Nf\|_{L^2}^2$. However, $\|[1-\chi_1]I\{P_N - Q_N\}T|_{im\, P_N}\|$ tends to zero since $[1-\chi_1]T$ is a compact and smoothing operator.
On the other hand, $P_NA\{P_N - I\}\chi I|_{im\, P_N} = 2P_NW_S\{P_N - I\}\chi I|_{im\, P_N}$ and $|\{P_N - I\}\chi IP_Nf(P)| \le o(1)\,|P_Nf(P)|$ with $o(1) \longrightarrow 0$ as $N \longrightarrow \infty$. Hence,

$$\|P_NW_S\{P_N - I\}\chi IP_Nf\| \le o(1)\|P_N^+W^+P_N^+|f|\,\|,$$

where $P_N^+f := \sum_{\iota \in \mathcal{J}} f(P_\iota)|\psi_\iota|$ and $\psi_\iota \in im\, P_N$ is defined by $\psi_\iota(P_\kappa) = \delta_{\iota,\kappa}$. Since $\|\,P_N^+|f|\,\| \le c\|P_Nf\|$ and the collocation operator $P_N^+W^+|_{im P_N^+}$ is bounded (repeat the arguments of part d)), the third term on the right-hand side of (4.10) tends to zero, too. Thus (4.10) implies iii) for the L^2 space. The supremum norm estimate is similar.
f) If we apply the just mentioned arguments to A_N from the modified method, we get two additional terms $Q_NT|_{im\, P_N}D_N$ and F_N, where $D_N := (d_{\iota,\kappa})_{\iota,\kappa}$, $d_{\iota,\kappa} := -[1 - v(P_\iota)]\delta_{\iota,\kappa}$ and $F_N := (f_{\iota,\kappa})_{\iota,\kappa}$ with $f_{\iota,\kappa} := (2W_S 1_N^P)(P_\iota)[\chi(P_\kappa) - \chi(P_\iota)]$ for $P_\iota, P_\kappa \in S^j$, $P_\iota \ne P_\kappa$ such that P_κ is a vertex of S and with $f_{\iota,\kappa} := 0$ else. However, multiplying by D_N from the right restricts the weakly singular operator T to a small neighborhood of the vertices of S. Thus $\|Q_NT|_{im\, P_N}D_N\|$ tends to zero for $N \longrightarrow \infty$. On the other hand, $|[\chi(P_\kappa) - \chi(P_\iota)]|$ is small if P_ι is close to P_κ and $|(2W_S1_N^P)(P_\iota)|$ is small if P_ι is far from P_κ. Hence, $|f_{\iota,\kappa}|$ is small and $\|F_N\| \longrightarrow 0$ for $N \longrightarrow \infty$. Consequently, iii) holds for the modified method, too.
g) Using the arguments of the proof of Lemma 3.2 ii) in [31], assertion ii) will follow analogously to part e). ∎

5 THE ASYMPTOTIC RATE OF CONVERGENCE

Throughout this section let us suppose that the right-hand side y of the boundary integral equation $Ax = y$ is continuous on S and a C^∞- function up to the boundary on each face of S. It is well known (cf. [25]) that, for any polyhedron, there exists a certain $\beta \in (0,1)$ such that $x = A^{-1}y$ is in the Hölder class $C^{0,\beta}$ for any such smooth y. Recall the definition of the parameter ζ and the meshsize h from (2.2). Then we get

THEOREM 5.1 *i) Let x_N denote the solution of the discretized collocation, i.e., x_N satisfies the equation (2.7), where the entry $a_{\kappa,\iota}$ is replaced by $a_{\kappa,\iota}^+$ defined in (2.13).*

Suppose this collocation method is stable (cf. Theorem 4.2 i)). Then there is a constant c independent of N such that

$$\|x - x_N\|_{L^2(S)} \leq c \max\{h^{\zeta/\log 4}\sqrt{\log h^{-1}}, h^4\}. \tag{5.1}$$

In particular, if the parameter ζ is greater than $8\log 2$, then we get the estimate $\|x - x_N\|_{L^2(S)} \leq c\, h^4$.

ii) Let x_N be as above the solution of the discretized collocation which is supposed to be stable, or let x_N denote the solution of the modified collocation, i.e., x_N satisfies the equation (2.16) and suppose this collocation method is stable (cf. Theorem 4.2 ii)). Then there is a constant c independent of N such that

$$\|x - x_N\|_{L^\infty(S)} \leq c \max\{h^{\beta\zeta/\log 2}, h^4\}. \tag{5.2}$$

In particular, if the parameter ζ is greater than $4\log 2/\beta$, then we get the estimate $\|x - x_N\|_{L^\infty(S)} \leq c\, h^4$.

Note that the number N of collocation points, i.e., the number of equations in (2.7) is of order $O(\, h^{-2} \log^2(h^{-1})\,)$. The computation of the stiffness matrix requires no more than $O(\, h^{-4} \log^4(h^{-1})\,)$ operations. Consequently, if we apply our discretized collocation (2.13) together with an iterative solver, we need $O(\, h^{-4} \log^4(h^{-1})\,)$ floating point operations. In order to prove the estimates of Theorem 5.1, let us start with two lemmata.

LEMMA 5.2 *For the exact solution $x = A^{-1}y$ and the coordinate transformation Φ^j of (2.1), we get*

$$\sup_{(s,t)\in D^j} |\partial_s^l (x \circ \Phi^j)(s,t)| \leq c, \ \sup_{(s,t)\in D^j} |\partial_t^l (x \circ \Phi^j)(s,t)| \leq c, \ l = 0, \ldots. \tag{5.3}$$

PROOF: It is sufficient to prove the assertion of the lemma for the case of the tangent cone $S = S_P$ of a corner point P (Note that the restriction of W_S acting between subsets of S with positive distance is a smoothing operator. Moreover, if we consider the neighborhood of an edge point, then the following arguments can also be applied if the Mellin transform is replaced by the Fourier transform.). First we suppose that y satisfies the estimates of x in (5.3) and we shall prove that $W_S y$ satisfies them, too. Without loss of generality let us consider S^j of type a) and introduce polar coordinates $\exp s = r \cos\varphi$ and $\exp s \exp t = r \sin\varphi$ over $\Phi^j(D^j) = S^j \subseteq S$. Then we get $\partial_s = r\,\partial_r$ and $\partial_t = \sin^2\varphi\, r\,\partial_r + \cos\varphi \sin\varphi\, \partial_\varphi$ and it remains to prove that the derivative $(r\,\partial_r)^k (\sin\varphi\,\partial_\varphi)^l W_S y$ is bounded. Let $V = \Phi^j(s,t) = P + r\cos\varphi\, \overrightarrow{PQ} + r\sin\varphi\, \overrightarrow{QR}$ and $U = \Phi^{j'}(s',t') = P + \rho\cos\psi\, \overrightarrow{PQ'} + \rho\sin\psi\, \overrightarrow{Q'R'}$ with $\exp s' = \rho \cos\psi$ and $\exp s' \exp t' = \rho \sin\psi$. For definiteness, we suppose that $Q = Q'$ and, for simplicity, that the angle between S^j and $S^{j'}$ is $\pi/2$ and that $|\, \overrightarrow{QR}\,| = |\, \overrightarrow{QR'}\,| = |\, \overrightarrow{PQ}\,| = 1$. Then we get $|U - V|^2 = (\rho\cos\psi - r\cos\varphi)^2 + r^2\sin^2\varphi + \rho^2\sin^2\psi$, $n_U \cdot (V - U) = r\sin\varphi$ and

$$\begin{aligned} W_S y(V) &= \frac{1}{4\pi}\int_0^\infty \int_0^{\pi/4} \frac{r \sin\varphi\, y(\rho,\psi)}{\sqrt{(r\cos\varphi - \rho\cos\psi)^2 + (\rho\sin\psi)^2 + (r\sin\varphi)^2}^3} d\psi \rho d\rho \\ &= \frac{1}{4\pi}\int_0^{\pi/4} \int_0^\infty \frac{r/\rho \sin\varphi\, y(\rho,\psi)}{\sqrt{(r/\rho)^2 + 1 - 2(r/\rho)[\cos\varphi\cos\psi]}^3} \frac{d\rho}{\rho} d\psi. \end{aligned} \tag{5.4}$$

Since the kernel takes the form of a Mellin convolution with respect to r and the differential operator $r\partial_r$ commutes with the Mellin convolution (cf. the proof of Theorem 3.1 in [17]), we conclude $(r\partial_r)^k W_S y = W_S(r\partial_r)^k y$ and it remains to prove the boundedness of $(\sin\varphi\,\partial_\varphi)^l W_S y$. However, if we denote the kernel of the integral operator by k, then it is not hard to show $|(\sin\varphi\,\partial_\varphi)^l k| \leq c|k|$. Thus the boundedness of the integral operator $W^+ \in \mathscr{L}(C(S))$ with the kernel function $|k|$ implies that $(\sin\varphi\,\partial_\varphi)^l W_S y$ is bounded. Now consider $x = A^{-1}y$. We get $A^{-1} = (I+2W_S)^{-1} = I - 2W_S A^{-1}$ and, using the fact that $(r\,\partial_r)$ commutes with W_S and A^{-1}, we conclude

$$(\sin\varphi\,\partial_\varphi)^l(r\,\partial_r)^k x = (\sin\varphi\,\partial_\varphi)^l(r\,\partial_r)^k y - 2(\sin\varphi\,\partial_\varphi)^l W_S A^{-1}(r\,\partial_r)^k y.$$

Obviously, the first term on the right-hand side is bounded. Moreover, $A^{-1}(r\,\partial_r)^k y$ is bounded and, again, $|(\sin\varphi\,\partial_\varphi)^l k| \leq c|k|$ implies that the second term on the right-hand side of the last equation is bounded. Thus the proof is finished. ∎

Let P_N denote the projection onto the span of $\{\varphi_\iota : \iota \in \mathscr{I}\}$ interpolating at $\{P_\iota : \iota \in \mathscr{I}\}$. Of course P_N takes the form $P_N f = \sum_{\iota\in\mathscr{I}} f(P_\iota)\psi_\iota$, where ψ_ι is in the span of $\{\varphi_\iota : \iota \in \mathscr{I}\}$ and $\psi_\iota(P_\kappa) = \delta_{\iota,\kappa}$. As mentioned at the beginning of the proof to Lemma 3.2, the functions ψ_ι decay exponentially. In particular, for an appropriate constant c_0 and $k_* := M_\infty - c_0 j_{lev}$ (cf. (2.2)), we get

$$|\psi_{(j,k,l)}(\Phi^j(s,t))| \leq c\,h^4 \tag{5.5}$$

if S^j is of type a) or c) and $t > s_{k_*}$, $l > M_\infty - 1$. We also get (5.5) if S^j is of type a) or b) and $s > s_{k_*}$, $k > M_\infty - 1$. Moreover, let $S(\chi)$ stand for the union of all sets $\{\Phi^j(s,t) : (s,t) \in D^j,\ s \leq s_{k_*}\}$ with S^j of type a) and b) and of all sets $\{\Phi^j(s,t) : (s,t) \in D^j,\ t \leq s_{k_*}\}$ with S^j of type a) and c). Let us denote the characteristic function of $S(\chi)$ by χ.

LEMMA 5.3 *Suppose that x is the solution of $Ax = y$. Then the following estimates hold*

$$\|x - P_N x\|_{L^2(S)} \leq c\,h^{\min\{\zeta/\log 4,4\}}, \qquad \|x - P_N x\|_{L^\infty(S)} \leq c\,h^{\min\{\beta\zeta/\log 2,4\}}, \tag{5.6}$$

$$\|\chi I(I - P_N)x\|_{L^2(S)} \leq c\,h^{\zeta/\log 4}, \qquad \|(1-\chi)I(I - P_N)x\|_{L^\infty(S)} \leq c\,h^4. \tag{5.7}$$

PROOF: Let us start with the second estimate of (5.6). By the arguments mentioned at the beginning of the proof to Lemma 3.2, we conclude that the P_N are uniformly bounded. Thus we only have to deduce the orders of approximation by functions from the space $im\,P_N$. In view of local quasi-interpolants (cf. [7]), it suffices to prove these rates of approximation locally. Now standard estimates show $O(h^4)$ convergence anywhere except in the vicinity of the edges, where we do not approximate by piecewise polynomials of degree three. In this vicinity of width $\exp(-[M_\infty - 2]h) \sim h^{\zeta/\log 2}$ we consider the function $x(P) = x(\Phi^j(s,t))$ with respect to the variable $\exp t$ and approximate this $C^{0,\beta}$- function by constants. This leads to an error of order $O(h^{\beta\zeta/\log 2})$.
In order to prove the first estimate of (5.6) it suffices to prove (5.7). Let us start with the first estimate of (5.7). The area of the support $supp\,\chi$ is less than $c\,h^{\zeta/\log 2}$ and we get

$$\|\chi I(I - P_N)x\|_{L^2(S)} \leq \|\chi\|_{L^2(S)}\,\|(I - P_N)x\|_{L^\infty(S)} \leq c\,h^{\zeta/(2\log 2)}.$$

On the other hand, over $supp\,(1-\chi)$ we interpolate by cubic tensor product splines and, in view of (5.5), the influence from the neighborhood of the edges is less than ch^4. Thus the order four estimate for usual cubic splines yields the second estimate of (5.7). ∎

PROOF: *(of Theorem 5.1)* Let $A_N \in \mathscr{L}(imP_N)$ stand for the discretized collocation operator, i.e., for the approximation of $P_N A|_{imP_N}$, where the entries $a_{\kappa,\iota}$ of the matrix corresponding to the bases $\{\varphi_\iota\}$ and $\{\psi_\iota\}$ are replaced by $a^+_{\kappa,\iota}$ of (2.13). We suppose A_N to be stable and show the estimates (5.1) and (5.2) for this method. Taking into account the stability, we obtain

$$\begin{aligned} x - x_N &= x - P_N x + A_N^{-1}\{A_N P_N x - A_N x_N\}, \\ \|x - x_N\|_{L^\infty} &\leq \|x - P_N x\|_{L^\infty} + c\,\|A_N P_N x - P_N A x\|_{L^\infty} \\ &\leq \|x - P_N x\|_{L^\infty} + c\{Te_1 + Te_2\}, \\ Te_1 &:= \|(A_N P_N - P_N A)\chi I x\|_{L^\infty}, Te_2 := \|(A_N P_N - P_N A)[1-\chi] I x\|_{L^\infty}. \end{aligned}$$

For the L^2 estimate, we get

$$\begin{aligned} \|x - x_N\|_{L^2} &\leq \|x - P_N x\|_{L^2} + c\{Te_2 + Te_3 + Te_4\}, \\ Te_3 &:= \|A_N P_N \chi I x\|_{L^2},\ Te_4 := \|P_N A \chi I x\|_{L^2}. \end{aligned}$$

In view of (5.6) it remains to estimate the terms Te_1, Te_2, Te_3, and Te_4. Using the uniform boundedness of A_N, the term Te_3 can be estimated analogously to the first inequality of (5.7). Let us turn to the estimation of Te_4. We obtain

$$\begin{aligned} Te_4^2 &\leq \sum_{\kappa\in\mathscr{I}} \|\psi_\kappa\|_{L^2}^2 |(A\chi x)(P_\kappa)|^2 \\ &\leq c \sum_{\kappa\in\mathscr{I},\,\chi(P_\kappa)=1} \|\psi_\kappa\|_{L^2}^2 \|x\|_{L^\infty}^2 + \sum_{\kappa\in\mathscr{I},\,\chi(P_\kappa)=0} \|\psi_\kappa\|_{L^2}^2 |(A\chi x)(P_\kappa)|^2. \end{aligned}$$

Using the exponential decay of the ψ_ι, we get $\|\psi_\kappa\|_{L^2} \sim \|\varphi_\kappa\|_{L^2}$ and the first sum on the right-hand side is less than $c\,\|\chi\|_{L^2}^2 \sim ch^{\zeta/\log 2}$. For the sake of definiteness, let us restrict the summation in the second sum to all $\kappa = (j,k,l)$ with a fixed j such that S^j is a type a) subdomain. Thus $\|\psi_\kappa\|_{L^2}^2 \sim \|\varphi_\kappa\|_{L^2}^2 \sim [\exp s_{k-1} - \exp s_k]\exp s_k[\exp s_{l-1} - \exp s_l]$. Since $W_S\chi(P_\kappa)$ is the solid angle under which $supp\,\chi$ is seen from P_κ, i.e., the angle under which a strip of width less than $\exp s_{k_*}$ is seen from a distance greater than $\exp s_k \exp s_l$, we get $|W_S\chi(P_\kappa)| \leq c\exp(s_{k_*} - s_k - s_l)$. Consequently, the second sum is less than

$$\begin{aligned} c \sum_{k,l=0,1,\ldots k_*} [e^{s_{k-1}} - e^{s_k}]\, e^{s_k}\, [e^{s_{l-1}} - e^{s_l}]\, e^{2(s_{k_*}-s_k-s_l)} &\leq c(1-e^{-h})^2 e^{s_{k_*}} \sum_{k,l=0,1,\ldots k_*} e^{s_{k_*}-s_l} \\ &\leq che^{s_{k_*}} M_\infty (1-e^{-h}) \sum_{m=0}^{\infty} e^{-mh} \leq ch^{-\zeta/\log 2}\log(h^{-1}). \end{aligned}$$

To estimate Te_2, we only have to estimate $|(A_N P_N - P_N A)[1-\chi] I x(P_\kappa)|$. For definiteness let us consider the integral and quadrature in $|(A_N P_N - P_N A)[1-\chi] I x(P_\kappa)|$ only over S^j of type a). Setting $Set := \{\Phi^j(s,t) : s_k^\& \leq s \leq s_{k-1}^\&,\ s_l^\# \leq t \leq s_{l-1}^\#\} \subseteq supp\,[1-\chi]$ and denoting the quadrature knots and weights of Simpson's rule (2.15) over Set by Q_i and θ_i, respectively, we obtain the following quadrature error over Set

$$Err := \left|\int_{Set} k(P_\kappa,V)[x(V)-x(P_\kappa)]d_V S - \sum_i k(P_\kappa,Q_i)[x(Q_i)-x(P_\kappa)]\theta_i\right| \le \tag{5.8}$$
$$|Set|\left\{\sup_{V\in Set}\left|\partial_s^4\left[k(P_\kappa,V)[x(V)-x(P_\kappa)]|D\Phi^j(s,t)|\right]\right|(s^\&_{k-1}-s^\&_k)^4\right.$$
$$\left.+\sup_{V\in Set}\left|\partial_t^4\left[k(P_\kappa,V)[x(V)-x(P_\kappa)]|D\Phi^j(s,t)|\right]\right|(s^\#_{l-1}-s^\#_l)^4\right\},$$

where $V=\Phi^j(s,t)$ and $P_\kappa=\Phi^{j_\kappa}(s_{k_\kappa},s_{l_\kappa})$. We observe $|s^\#_{l-1}-s^\#_l|\le ch$ and $|s^\&_{k-1}-s^\&_k|\le c\exp(-s^\&_k)|\exp s^\&_k-\exp s_{k_\kappa}|h$. For the kernel k of W_S, it is not hard to derive (cf. the proof of Lemma 5.2) that $|\partial_s^l k(P_\kappa,\Phi^j(s,t))|\le c|k(P_\kappa,\Phi^j(s,t))|\exp(ls)/|\exp s_{k_\kappa}-\exp s|^l$ and $|\partial_t^l k|\le c|k|$, $l=0,\dots,4$. Moreover, from Lemma 5.2 we infer that $|\partial_s^l x|\le c$ and that $|\partial_t^l x|\le c$. For an appropriate point $P'\in Set$, we obtain $Err\le c|Set|\,|k(P_\kappa,P')|\,h^4$. Using (4.5) and summing up over all sets Set leads to

$$\begin{aligned}|(A_NP_N-P_NA)[1-\chi]Ix(P_\kappa)| &\le \sum Err \le ch^4\sum|Set|\,|k(P_\kappa,P')|\\ &\le ch^4\int_S|k(P_\kappa,\cdot)|\le c\,h^4.\end{aligned}$$

For the estimation of Te_1, we have to consider Err of (5.8) over a set $Set\subsetneq supp\,\chi$. Taking into account that $x\in C^{0,\beta}$, we conclude

$$Err\le c\int_{Set}|k(P_\kappa,V)|\,|V-P_\kappa|^\beta d_VS,\quad \sum_{Set\subsetneq supp\,\chi} Err\le c\int_{supp\,\chi}|V-P_\kappa|^{\beta-2}d_VS. \tag{5.9}$$

For the sake of definiteness, let us estimate the last integral over S^j of type a) and suppose $P_\kappa\in S^{j'}$. Moreover, we suppose that all the assumptions leading to (5.4) are satisfied. Set $P_\kappa=\Phi^{j'}(s_{k'},s_{l'})$ and, since the kernel function k vanishes over S^j, suppose $l'<M_\infty$. We get

$$\begin{aligned}\int_{supp\,\chi\cap S^j}|V-P_\kappa|^{\beta-2}d_VS &\le \int_{-\infty}^{s_{k_*}}\int_{-\infty}^0\left|(e^{s_{k'}}-e^s)^2+e^{2s_{k'}}e^{2s_{l'}}+e^{2s}e^{2t}\right|^{\frac{\beta-2}{2}}e^{2s}e^tdsdt,\\ &\le c\int_0^1\int_0^{se^{s_{k_*}}}\left||e^{s_{k'}}-s|+e^{s_{k'}+s_{l'}}+t\right|^{\beta-2}dtds.\end{aligned}$$

Performing the integrations, we arrive at the upper bound

$$\begin{aligned}&\frac{c}{(\beta-1)\beta}\left\{\left|e^{s_{k'}}e^{s_{k_*}}+e^{s_{k'}+s_{l'}}\right|^\beta\frac{-2}{1-e^{2s_{k_*}}}+\left|e^{s_{k'}}+e^{s_{k'}+s_{l'}}\right|^\beta\frac{e^{s_{k_*}}}{1-e^{s_{k_*}}}+\right.\\ &\left.2\left|e^{s_{k'}+s_{l'}}\right|^\beta-\left|1-e^{s_{k'}}+e^{s_{k'}+s_{l'}}\right|^\beta+\left|1+e^{s_{k_*}}-e^{s_{k'}}+e^{s_{k'}+s_{l'}}\right|^\beta\frac{1}{1+e^{s_{k_*}}}\right\}\\ &\le O(e^{s_{k_*}})+c\left\{\left|\left|1+e^{s_{k_*}}-e^{s_{k'}}+e^{s_{k'}+s_{l'}}\right|^\beta-\left|1-e^{s_{k'}}+e^{s_{k'}+s_{l'}}\right|^\beta\right|+\right.\\ &\left.\left|\left|e^{s_{k'}}e^{s_{k_*}}+e^{s_{k'}+s_{l'}}\right|^\beta-\left|e^{s_{k'}+s_{l'}}\right|^\beta\right|\right\}.\end{aligned}$$

The last expression is bounded by $c\exp s_{k_*}(\exp s_{M_\infty-1})^{\beta-1}\le ch^{\beta\zeta/\log 2}$.

Finally, let us consider the modified method (2.16). The error of this algorithm can be estimated analogously to the discretized collocation. The only additional term to be controlled is $(W_S x_N^P)(P_\kappa) - x_N(P)(W_S 1_N^P)(P_\kappa)$ for a vertex $P = P_{\iota_0}$. By the definition of x_N^P and 1_N^P we conclude

$$\begin{aligned}(W_S x_N^P)(P_\kappa) - x_N(P)(W_S 1_N^P)(P_\kappa) &= \sum_{\iota\in\mathscr{I}^P}[1 - v(P_\iota)][\xi_\iota - \xi_{\iota_0}](W_S\varphi_\iota)(P_\kappa)\\ &\leq c \sup_{\iota:[1-v(P_\iota)]>0} |\xi_\iota - \xi_{\iota_0}|.\end{aligned}$$

However, $[1 - v(P_{(j,k,l)})] > 0$ implies $k \geq M_\infty - c_1 2^{j_{lev}}$ for an appropriate constant c_1. Hence, $\exp s_k \leq c2^{\zeta/\log 2}$ and the Hölder condition implies $|x(P_\iota) - x(P_{\iota_0})| \leq c2^{\beta\zeta/\log 2}$. Now we observe that $\xi = (\xi_\iota)_{\iota\in\mathscr{I}} = (\varphi_\iota(P_\kappa))^{-1}_{\kappa,\iota\in\mathscr{I}}(x(P_\iota))_{\iota\in\mathscr{I}}$, where the entries of $(\varphi_\iota(P_\kappa))^{-1}_{\kappa,\iota\in\mathscr{I}}$ have exponential decay (cf. the beginning of the proof to Lemma 3.2). From this we infer $|\xi_\iota - \xi_{\iota_0}| \leq c2^{\beta\zeta/\log 2}$ for $\iota \in \mathscr{I}^P$ and $[1 - v(P_\iota)] > 0$. ■

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